THE
PHYSICAL
SOCIETY

MEMOIRS

OF THE

COLLEGE OF SCIENCE

KYOTO IMPERIAL UNIVERSITY

VOL. II, NO. 3

JULY 1917

KYOTO

On the Solubility of Lead Acetate in Water.

By

Yukichi Osaka and Reijiro Hara.

(Received January 15, 1917.)

When lead acetate is dissolved in water, the solution is not clear owing to its hydrolysis, and any systematic determination of its solubility is not to be found in literature. The Landolt-Börnstein's *Tabellen* gives no data for it, in *A.* Seidelle's *Solubilities of Inorganic and Organic Substances*[1] a single datum for 25° is given, as 50 grams in 100 grams water, while in the *Tables annuelles de Constantes et Données numeriques de Chimie, de Physique et de Technologie*[2] it is reported as 72·5 instead of 50.

Led by these facts, we have undertaken to study the solubility of lead acetate in water. We have not determined its actual solubility, but estimated its solubility in pure water if no hydrolysis had taken place, by extrapolation from its solubilities in dilute solution of acetic acid.

Commercially pure lead acetate was twice recrystallised, and the final product was found to be pure to 99·96 per cent. by analysical determination of its lead. As solvents, 0·10, 0·05, and 0·025 normal acetic acid were used. The experiments were carried out in the following way.

The solvent was put in a small Erlenmeyer flask which was well steamed, an excess of lead acetate was added and the well-stoppered flask was made to rotate in a thermostat for two days and nights, then it was allowed to stand still. When the residue had subsided, a portion of the solution was transferred to a weighed bottle by a warmed pipette with a tube filled with some refined cotton wool, attached to its end, and the bottle was weighed again. The sample taken out was diluted to 100 cc., and 20 cc. of the diluted solution

---

1 p. 162.

2 **2**, 462: *from* Hyg. Lab. U. S. Publ. Health and Mar.-Hosp. Serv.. Bull., **67**, 91.

was used for a single analysis. In a few cases the original solutions were directly subjected to analysis. The lead was determined as the sulphate in the usual way.

The method of calculation may be best illustrated by an example.

The original solution taken for analysis in grams		1·1083
Lead in it		0·2452
Lead acetate ($(C_2H_3O_2)_2Pb$) corresponding to it		0·3849
Water of crystallisation originally contained in the acetate dissolved		0·0644
Solvent in the sample taken		0·6590
Acetic acid in it		0·0040
Water in it		0·6550
Total water in the sample		0·7194
Percentage composition of the sample	lead acetate	34·73
	water	64·91
	acetic acid	0·36
Percentage composition, excluding acetic acid	lead acetate	34·86
	water	65·14
Lead acetate in 100 grams water		53·52

The results calculated in this way are given in the following table:

Normality of acetic acid used as solvent.	No.	Sample analysed.	Lead found.	Percentage composition of the sample.			Lead acetate in 100 grms of water.
				$(C_2H_3O_2)_2Pb$	$C_2H_4O_2$	H_2O	
Temperature: 25·0°							
0·10	1	1·1083	0·2452	34·73	64·91	0·36	53·52
	2	0·5272	0·1162	34·69	64·95	0·36	53·49
	3	0·8241	0·1823	34·73	64·92	0·35	53·48
	4	1·1022	0·2440	34·75	64·90	0·35	53·54
							53·51
0·05	1	0·6555	0·1462	35·01	64·81	0·18	54·02
	2	0·6755	0·1506	34·98	64·83	0·19	53·97
	3	0·7124	0·1585	34·92	64·88	0·20	53·90
	4	0·7579	0·1686	34·93	64·90	0·17	53·81
							53·92
0·025	1	0·3884	0·0869	35·11	64·82	0·070	54·17
	2	0·3884	0·0809	35·11	64·82	0·070	54·17
							54·17

Normality of acetic acid used as solvent.	No.	Sample analysed	Lead found.	Percentage composition of the sample.			Lead acetate in 100 grm. of water.
				$(C_2H_3O_2)_2Pb$	$C_2H_4O_2$	H_2O	
Temperature: 35·0°							
0·10	1	0·8921	0·2635	46·34	53·36	0·28	86·88
	2	0·8921	0·2632	46·32	53·40	0·28	86·73
	3	0·5246	0·1549	46·36	53·36	0·29	86·90
	4	0·5246	0·1550	46·38	53·34	0·29	86·95
							86·87
0·05	1	0·5692	0·1689	46·56	53·30	0·14	87·34
	2	0·5692	0·1688	46·54	53·32	0·14	87·24
	3	0·8108	0·2405	46·56	53·31	0·14	87·34
	4	0·8108	0·2406	46·58	53·28	0·14	87·43
							87·34
0·025	1	0·4665	0·1386	46·65	53·29	0·060	87·53
	2	0·4934	0·1467	46·66	53·28	0·060	87·56
							87·55
Temperature: 45·0°							
0·110	1	0·6409	0·2467	60·43	39·40	0·17	153·39
	2	0·6409	0·2466	60·40	39·43	0·17	153·18
	3	0·8235	0·3170	60·43	39·39	0·18	153·40
	4	0·8235	0·3170	60·43	39·39	0·18	153·40
							153·34
0·05	1	0·6263	0·2415	60·56	39·36	0·080	153·87
	2	0·6263	0·2416	60·55	39·37	0·080	153·75
	3	0·9388	0·3620	60·54	39·38	0·085	153·71
	4	0·9388	0·3620	60·54	39·38	0·085	153·71
							153·76
0·025	1	0·5598	0·2162	60·61	39·35	0·04	154·01
	2	0·5772	0·2221	60·61	39·34	0·05	154·05
							154·03

If we represent graphically the solubility of lead acetate as a function of the normalities of acetic acid, it is found that it is linear, and we have extrapolated the solubility of lead acetate in pure water analytically, by the method of the least square, and has obtained the following results :

Temperature.	Solubility of lead acetate in 100 grams water.
25·0°	54·38
35·0	87·77
45·0	154·25

These are naturally the would-be solubility when no hydrolysis would take place.

Graphical Solution of $\frac{dy}{dx} = f(x, y)$.

By

Takeo Wada.

(Received February 2, 1917.)

The problem of this paper is to investigate the integral curves defined by differential equations of the simplest form

$$\frac{dy}{dx} = f(x, y)$$

in which $f(x, y)$ is a rational function of x and y.

Equations of this form occur frequently in applied mathematics, and were first treated by Profs. Briot and Bouquet[1]. Subsequently Prof. Poincaré[2] gave a complete and elegant treatment. Both, however, being based upon the theory of analytic functions of complex variables, are a little complicated.

I have tried to consider the problem from the standpoint of real variables, and, by the help of the ideas of these professors, especially Prof. Poincaré's idea of consecutive point, to obtain a method of treatment. In principle the endeavour is to shew the outline of integral curves by the use of slope and concavity, just as is done in tracing curves defined by equations between x and y.

1. General Theory of $\frac{dy}{dx} = f(x, y)$.

By the famous theorem of Cauchy-Lipschitz it is known that, if $f(x, y)$ is continuous and $\frac{\partial f}{\partial y}$ is limited in the neighbourhood of a point x_0, y_0, there exists one integral of the differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

which passes through the point x_0, y_0.

---

1 J. d l'Ecole Polytech. **36**, 133-198 (1856).

2 Sur les courbes définies par les équation differentielle (J. Liouville 1881, 1882 and 1885).

In particular, when $f(x,y)$ is a rational function (1) may be written in the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{2}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials of x and y, which have no common factor; and we shall treat hereafter the differential equations of this form.

In equation (2), the existence-theorem applies at any point where $Q(x,y)$ does not vanish. Though at a point x_0, y_0 $f(x,y)$ vanishes, if $P(x_0,y_0) \neq 0$, considering y as the independent variable, the differential equation

$$\frac{dx}{dy} = \frac{Q(x,y)}{P(x,y)}$$

satisfies the Cauchy-Lipschitz's condition at x_0, y_0; and the existence of a single integral curve through the point is assured. Thus passing through any point where $P(x,y)$, $Q(x,y)$ do not vanish at the same time there is one integral curve of (2).

The points where $P(x,y)$ and $Q(x,y)$ do not vanish at the same time are called *ordinary points*; and the exceptional points are called *singular points* of the differential equation (2). The singular points are given by the simultaneous equations

$$P(x,y) = 0, \quad Q(x,y) = 0 \tag{3}$$

and there is a finite number of such points in the plane.

Denoting the integral curve which passes through an ordinary point by $C(x_0,y_0)$, it may be expressed analytically by

$$C(x_0,y_0): \begin{cases} x = \varphi(t\,;\, x_0, y_0), \\ y = \psi(t\,;\, x_0, y_0) \end{cases} \tag{4}$$

where φ and ψ are one-valued functions which are continuous with the derivatives φ_t', ψ_t' and satisfy the equation

$$\frac{\psi_t'}{\varphi_t'} = f(\varphi, \psi)$$

in a certain interval (T_1, T_2) of t and the initial conditions

$$\varphi(t_0\,;\, x_0, y_0) = x_0, \quad \psi(t_0\,;\, x_0, y_0) = y_0$$

t_0 being a fixed value in (T_1, T_2).

The functions φ, ψ are continuous not only of t but also of the parameters[1] x_0, y_0 in any domain $D(t, \xi, \eta)$ which contains $t=t_0, \xi=x_0, \eta=y_0$ and does not contain any singular point, provided that for any set of values t, ξ, η in D the functions $\varphi(t; \xi, \eta), \psi(t; \xi, \eta)$ do not represent any singular point. This means that the integral curve moves continuously with the initial point x_0, y_0.

From the continuity of φ_t' and ψ_t' it follows that the integral curve $C(x_0, y_0)$ is rectifiable. If we take the arc length s measured from x_0, y_0 on a direction as parameter, the equations to the curve will become

$$(4)_a \qquad C(x_0, y_0): \quad x=\varphi(s; x_0, y_0),\ y=\psi(s; x_0, y_0) \quad s_1 \leqq s \leqq s_2$$

of which we have

$$\frac{\psi_s'}{\varphi_s'} = f(\varphi, \psi), \quad (\varphi_s')^2+(\psi_s')^2 = 1$$

and

$$\varphi(0; x_0, y_0) = x_0, \quad \psi(0; x_0, y_0) = y_0.$$

As an integral curve can not stop at any ordinary point, it may be extended on both sides so long as it does not arrive at a singular point. The curve so extended as possible on both sides is called a *complete characteristic.* Of characteristics the following properties are reckoned:

1) They can not have any singular point in the interior, and their endpoints, if they exist, are singular;

2) They can not have any branch-point in the interior;

3) No two characteristics can have a point in common, which is not an endpoint of both;

4) The segment between any two interior points of a characteristic has a finite length.

That form of characteristic which is constructed through an ordinary point by possible extension on one side of the point is called a *semi-characteristic.* Assuming a direction, two semi-characteristics through an ordinary point are distinguished as progressive and regressive.

Of a semi-characteristic the following four forms are to be considered:

i) The continuation may finally arrive at a singular point and stop at that point;

ii) It may return again to the point of starting. In this case the curve is closed and is called a *cyclic characteristic.*;

---

1 E. Picard, Traite d'analyse, II, p. 333. (1905).

iii) It may go to an infinite distance;

iv) The semi-characteristic may have no second endpoint, however we may continue the extension, the curve itself remaining in a finite domain. This form will be considered later.

2. Slope and Concavity defined by a Differential Equation.

In tracing a curve defined by an equation between x and y, the general principle is due to the investigation of 1) the existence of the curve, 2) $\frac{dy}{dx}$ i.e. the slope of the curve, 3) $\frac{d^2y}{dx^2}$ i.e. the concavity of the curve, 4) the nature of multiple points and 5) the forms of indefinitely distant branches. This consideration is directly applied also to the discussion of curves defined by a differential equation.

In the first place we shall consider the slope of characteristics defined by

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} = f(x,y). \tag{2}$$

Divide the points of the entire plane into five aggregates, according as the value of $f(x,y)$ is I positive, II negative, III zero, IV infinite or V indeterminate, then

at points of I the line-elements are increasing,
" " " II " " " " decreasing,
" " " III " " " " parallel to the x-axis,
" " " IV " " " " " " " " y-axis

and the points of V are singular points of the differential equation. Aggregates I and II consist of points in one or more domains, III and IV of points on curves and isolated points and V of a finite number of points.

Next to find the concavity of the characteristics from the equation (2), deduce the second derivative;

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y) = \frac{Q^2P_x' + PQ(P_y' - Q_x') - P^2Q_y'}{Q^3} \tag{5}$$

and, denoting the expression on the right of (5) by $F(x,y)$, we have

$$\frac{d^2y}{dx^2} = F(x,y), \tag{5}_a$$

where the function $F(x,y)$ is evidently rational of x and y.

Again divide the points in the plane into aggregates, according as the value of $F(x,y)$ at x,y is VI positive, VII negative or VIII zero. Then

at points of VI the characteristics are concave upwards,
„ „ „ VII „ „ „ „ downwards
and „ „ „ VIII „ „ have contact of higher

order than the first, with tangent lines at those points. For example, a point of VIII may be an inflexional point of the characteristic through that point.

By the above two steps the entire plane will have been divided into the following aggregates of points :

a) Four kinds of domains where

i) $f>$, $F>0$, ii) $f>0$, $F<0$, iii) $f<0$, $F<0$, iv) $f<0$, $F<0$;

b) Three kinds of curves where

v) (6) $f(x,y) = 0$ or $P(x,y) = 0$,
vi) (7) $f(x,y) = \infty$ or $Q(x,y) = 0$,
vii) (8) $F(x,y) = 0$ or $Q^2P_x' + PQ(P_y' - Q_x') - P^2Q_y' = 0$;

c) viii) Finite number of singular points where P and Q vanish at the same time.

When the configuration of slope and concavity is known, we may trace the outline of characteristics excluded the neighbourhoods of the singular points.

In tracing the characteristics it is important to know how the curve (8) cuts the characteristics. Denoting by x_1, y_1 a point on (8), at that point the slope of this curve is $\left(-\frac{F_x'}{F_y'}\right)_{x_1,y_1}$ while that of the characteristic is $f(x_1,y_1)$. Hence if

$$(9) \qquad \chi_{F=0} = \frac{dy_1}{dx_1} - f(x_1,y_1) = -\left\{\frac{F_x' + F_y' f}{F_y'}\right\}_{x_1,y_1}$$

is positive the characteristic, in passing through the point (x_1,y_1) from the left to the right, cuts $F=0$ from the upper side to the lower; if it is negative, from the lower to the upper, and, finally, if it is zero, the curve (8) and the characteristic have the same line-elements at the point (x_1,y_1).

3. Direction Angles and Characteristic Angles.

Before going into the consideration of characteristics in the neighbourhood of a singular point, certain preliminary conceptions should be introduced.

Regular curve. By a regular (open) curve is meant an unclosed Jordan curve which has a tangent line everywhere, that moves continuously with the point on the curve.

The analytical expression of a regular curve is

$$x = \Phi(t), \qquad y = \Psi(t) \qquad T_1 \leqq t \leqq T_2 \tag{10}$$

in which $\Phi(t)$, $\Psi(t)$ aud their derivatives $\Phi'(t)$, $\Psi'(t)$ are continuous and satisfy the conditions

$$\{\Phi'(t)\}^2 + \{\Psi'(t)\}^2 > 0 \qquad T_1 \leqq t \leqq T_2,$$

and

$$\{\Phi(t_1) - \Phi(t_2)\}^2 + \{\Psi(t_1) - \Psi(t_2)\}^2 > 0$$

for any two different values t_1, t_2 in (T_1, T_2).

Directed line. A directed line is a straight line which has a fixed positive sense. Of a directed line two sides are distinguished, namely the right and the left sides.

Let there be two directed lines L_0 and L. Assuming the positive sense for measuring the angle counterclockwise, the angle between the two lines may be determined up to a multiple of 2π. This ambiguity is removed when multiples of 2π are left out of account; and then the measure is expressed by a value between 0 and 2π. The angle between L_0 and L lies in $(0, \pi)$ when L is directed leftwards; otherwise, the angle lies in $(\pi, 2\pi)$.

The angle between the positive x-axis and a directed line L is called the *direction angle* of the line L.

The conception of senses, sides and direction angles is also made use of in connection with regular curves and line-elements.

To apply these to the line-elements defined by a differential equation of the form (2), take an ordinary point $M_0(x_0, y_0)$, and for the direction angle at that point assign a fixed one, say α_0, of the angles which satisfy

$$\cos\alpha = \frac{\sigma\, Q(x_0, y_0)}{\sqrt{\{P(x_0, y_0)\}^2 + \{Q(x_0, y_0)\}^2}}, \quad \sin\alpha = \frac{\sigma\, P(x_0, y_0)}{\sqrt{\{P(x_0, y_0)\}^2 + \{Q(x_0, y_0)\}^2}}$$

in which for σ one of the values 1 or -1 is to be given at will. To

define the direction angle at an arbitrary ordinary point $M_1(x_1, y_1)$, assume a curve γ_1 connecting M_0 and M_1, which does not contain any singular point of the differential equation. Then the angle α defined by

$$(11) \qquad \cos\alpha = \frac{\sigma Q(x,y)}{\sqrt{P^2+Q^2}}, \quad \sin\alpha = \frac{\sigma P(x,y)}{\sqrt{P^2+Q^2}}$$

varies continuously with the point $M(x, y)$ on the curve γ_1, starting from the initial value α_0. The final angle α_1 at M_1 is defined as the *complete direction angle* at M_1.

If instead of γ_1 another curve γ_2 which connects the same points M_0, M_1 is taken, the corresponding complete direction angle at M_1 will be determined. But since $\cos\alpha_1 = \cos\alpha_2$, $\sin\alpha_1 = \sin\alpha_2$ the two complete direction angles α_1, α_2 differ at most by a multiple of 2π. Thus the complete angle α at a point $M(x, y)$ depends not only upon the coordinates x, y but also upon the curve which combines the point with the initial point M_0.

To make the direction angle a one-valued function of the coordinates we have only to leave any integral multiple of 2π out of consideration. This measure of the direction angle is called the *simple direction angle.*

Generally if two ordinary points N_1, N_2, of which the complete angle at N_1 is given, be connected by two curves γ', γ'', two corresponding direction angles at N_2 are given immediately. The difference of these angles is an integral multiple of 2π; and is the change of the complete direction angle when the line-element makes a complete revolution moving along the closed curve γ formed by the two curves γ', γ'' (Fig. 1). It is clear that the difference depends upon the form of the closed curve γ and the sense of revolution but not upon the starting point. Assuming the sense of revolution counter-clockwise, the change of the complete direction angle, which arises from a complete revolution along γ is called the *characteristic angle of the closed curve* γ. It is an integral multiple of 2π.

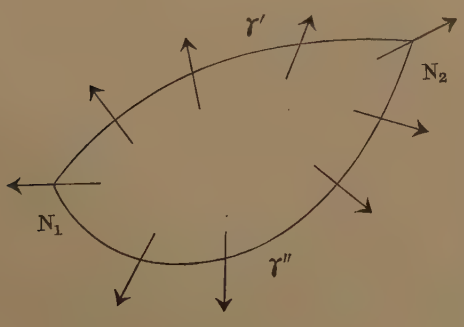


Fig. 1.

Suppose within and on a closed curve γ there is no singular point, then the complete direction angle α defined by (11) is one-valued and continuous within and on the closed curve γ. The characteristic angle of the cycle is therefore zero. Thus we have:

Theorem 1. If the characteristic angle of a closed curve is not zero, in the interior of the curve there exists at least one singular point.

As the characteristic angle of a cyclic characteristic is not zero, being 2π, it follows:

Cor. In the interior of a cyclic characteristic, there exists at least one singular point.

Let γ_1, γ_2 be any two closed curves which contain only one singular point S in the interior, then it is easily shewn that the characteristic angles of γ_1 and γ_2 are equal. Hence the characteristic angle of such a closed curve is proper to the point S, and is called the *characteristic angle of the singular point* S.

Without difficulty we may further conclude that the characteristic angle of any closed curve is the sum of the characteristic angles belonging to the singular points within the closed curve.

4. Consecutive and Contact Points.[1]

Let two regular curves γ_1 and γ_2 intersect at only two points M_1, M_2, then the two segments of the curves intercepted between the points, will form a closed curve which divides the entire plane into two parts, interior and exterior domains.

The form of the intersection of the curves is of two types: Type I, in which all the four extensions of the segments of γ_1, γ_2 at the ends M_1, M_2 lie in one and the same domain, exterior (Fig. 2) or interior (Fig. 3), Type II in which the two extensions of the segments at M_1 lie in one domain while those at the second endpoint M_2 lie in the other domain (Fig. 4).

In a plane where directed line-elements are defined by (11), consider a regular directed curve

$$\gamma: \quad x = \Phi(t),\ y = \Psi(t) \qquad T_1 \leqq t \leqq T_2$$

which passes none of the singular points of the differential equation $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$.

[1] J. Liouville, p. 409 (1881),

Fig. 2.

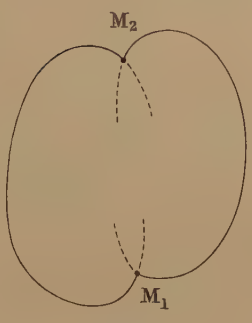


Fig. 3.

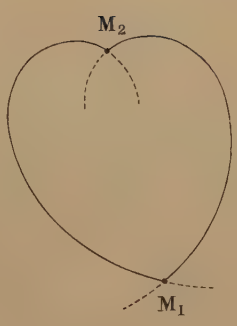


Fig. 4.

Take a point $M(t)$ on the curve γ and let $C(t)$ be the progressive semi-characteristic through the point M. This may be expressed by

$$C(t): \quad x = X(s, t), \; y = Y(s, t) \qquad s \geqq 0,$$

where s is the arc length measured from M on the positive direction and the functions X, Y satisfy the relations

$$(X_s')^2 + (Y_s')^2 = 1,$$
$$X(0, t) = \Phi(t), \quad Y(0, t) = \Psi(t).$$

When γ and $C(t)$ touch each other at M, the curve γ is said to have a *contact* at the point M. In particular the point M is called a contact point of the first kind when γ is cut by $C(t)$ at M, and of the second kind when, in the neighbourhood of M, $C(t)$ lies on one side of γ.

When these two curves have other common points besides M, if in a neighbourhood of M, γ and $C(t)$ have no other common point, on the curve $C(t)$ there will be the first point N which is common to γ. The point N is called the *consecutive point* of M with regard to the curve γ. When, instead of the progressive semi-characteristic, the regressive semi-characteristic $C_2(t)$ through M is considered, the first point on $C_2(t)$ which is common with γ is called the *preceeding point* of M.

Suppose that a point M on the curve γ has the consecutive point N and the two curves γ and $C(t)$ do not touch at M and N, then the two segments of the curves will have an intersection of the type I or II. Accordingly the point N is called the consecutive point of the *first* or *second kind*.

We can now prove:

Theorem 2. If a point M on a regular curve γ which does not contain any singular point of a differential equation $\frac{dy}{dx}=f(x,y)$ have the consecutive point N of the first kind, then on the segment of γ between M, N there exists at least one contact point.

Proof. Let ds be the directed line-element defined by

$$\cos\alpha=\frac{Q}{\sqrt{P^2+Q^2}},\quad \sin\alpha=\frac{P}{\sqrt{P^2+Q^2}}.$$

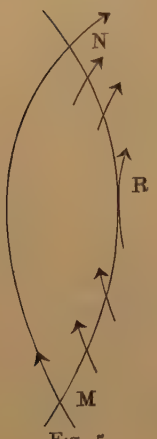


Fig. 5.

To make γ a directed curve assign a positive sense. Suppose at M ds is directed leftwards of γ, then as N is the consecutive point of the first kind, at that point ds is directed rightwards of γ. Denote by θ the simple measure of the angle between the curve γ and the line-element ds on γ, then

$$0<\theta_M<\pi, \text{ at } M \text{ and } \pi<\theta_N<2\pi \text{ at } N.$$

As the angle θ varies continuously from θ_M to θ_N when ds moves along γ, it must take the value 0 or π at a certain point R of the segment γ_{MN}. R is a point of contact of the curve γ.

5. Applications of the Theory of Contact Points.

There are numerous applications of the theory of contact points to the differential equation (2).

Theorem 3. Any algebraic curve which is not a characteristic of a differential equation (2) can not have indefinitely many contact points.

Proof. Let γ_A be the curve defined by an irreducible algebraic equation $A(x,y)=0$. The contact points of this curve are then given by the simultaneous equations

$$A=0,\quad Q\frac{\partial A}{\partial x}+P\frac{\partial A}{\partial y}=0.$$

As these two equations are algebraic there is a finite number of solutions. Specially if $A(x,y)$ is of the m^{th} degree and $f(x,y)$ is of the n^{th} degree, the curve γ_A may have at most $m\{(m-1)+n\}$ contact points.

Theorem 4. If a semi-characteristic of (2) has indefinitely many

common points with an algebraic curve which is not a characteristic, it is a spiral.

Proof. Let C be a semi-characteristic which has indefinitely many points in common with an algebraic curve γ_A. As γ_A may have a finite number of branch points and singular points of the differential equation, we may determine such a segment γ_{NK}, that has 1) indefinitely many points, say $\{M\}$, belonging to C, of which N is a limiting point; 2) no branch point, no singular point and no contact point in the interior.

Take a point M_1 of $\{M\}$, then on the arc γ_{M_1N} there is necessarily a consecutive or preceeding point of M_1. To fix the idea, suppose that there is a consecutive point of M_1, call it M_2, then since γ_{M_1N} contains no contact point, by theorem 2, M_2 is a consecutive point of the second kind. We know also that the arc $\gamma_{M_1M_2}$ besides M_1, M_2 contains no other point belonging to $\{M\}$. It is also clear that the point M_2 has a consecutive point of the second kind with regard to the curve γ_{NK}, call it M_3. M_3 lies in γ_{M_2N}. Similarly M_3 has a consecutive point M_4 on the arc γ_{M_3N}; and so on.

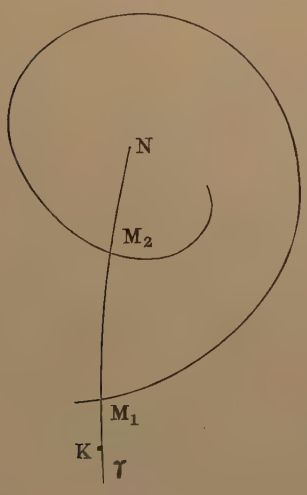


Fig. 6.

Thus proceeding we obtain an infinite sequence of points $M_1, M_2, \ldots M_i, M_{i+1}, \ldots$ converging to N, of which each is the consecutive point of the second kind of the just preceeding. Thus it is shewn that the characteristic C is really a spiral.

As each segment CM_iM_{i+1} of the characteristic C lies on the same sides of the preceeding segments CM_jM_{j+1} $(j=1, 2, \ldots i-1)$, the arc CM_iM_{i+1} will converge to the point N or to a curve C_L through N.

In the first case the point N is a singular point, for each cycle λ_i which consists of the segments CM_iM_{i+1} and $\gamma_{M_iM_{i+1}}$ contains N and the characteristic angle of λ_i is 2π (§ 3).

In the second case the limiting curve C_L is a closed one. For, as the consecutive point of M_i is M_{i+1} and $\lim_{i\to\infty} M_i = \lim_{i\to\infty} M_{i+1} = N$, the curve C_L, which is the limiting position of CM_iM_{i+1}, starting from the point N must return to that point. In the neighbourhood of a

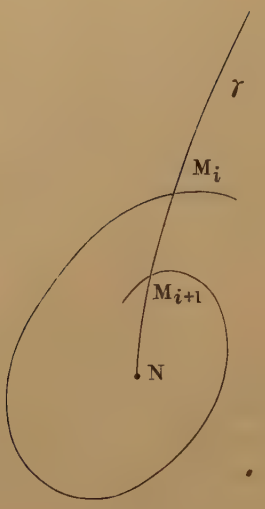


Fig. 7.

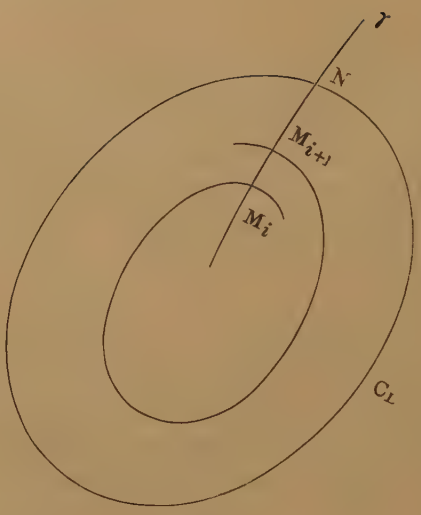


Fig. 8.

segment of C_L, which does not contain any singular point, the curves $CM_i\,M_{i+1}$ converge uniformly and therefore the line-element of C_L at any ordinary point satisfies the differential equation. This shews that C_L is an integral curve. We shall call the curve C_L, following Prof. Poincaré, a limiting cycle. When C_L does not pass through any singular point, it is a cyclic characteristic, but when it passes through singular points it may consist of a finite number of loops and a finite number of characteristics connecting two of the singular points as shewn in the figure.[1]

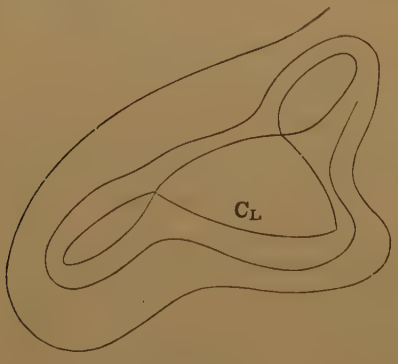


Fig. 9.

We have thus established:

Theorem 5. Any semi-characteristic of spiral form converges to a limiting cycle or to a singular point.

Theorem 6. Any semi-characteristic which has an infinite number of extreme values of x or y, or indefinitely many inflexional points is a spiral.

Proof. That a semi-characteristic C has an infinite number of extreme values of y means that the two curves C and $P(x,y)=0$ have

[1] See p. 180.

indefinitely many points in common. Hence, by theorem 4, C is a spiral. The other cases are proved similarly.

We can now determine the form of the semi-characteristic (iv) introduced in § 1.

Theorem 7. A semi-characteristic which does not go to an infinite distance or to any singular point is a spiral converging to a limiting cycle.

Proof. Had the semi-characteristic a limited arc-length, it must converge to a limiting point which is necessarily a singular point. As this case is rejected the curve must have an infinite arc-length. Hence it follows that the characteristic has an infinite number of extreme values of x or y; and thus it must be a spiral converging to a limiting cycle.

6. Possible Directions at Singular Points.

The first question which arises in the discussion of characteristics near a singular point S is whether there will be a characteristic which passes through the point S and has a definite direction at that point.

As the first step we investigate the directions along which there may exist an integral curve. Such directions are called *possible directions* at the singular point S.

For the purpose transform the origin to the singular point, then we have

$$\frac{dy}{dx} = \frac{P_1(x, y)}{Q_1(x, y)} \tag{12}$$

where P_1 and Q_1 are polynomials which satisfy

$$P_1(0, 0) = 0, \quad Q_1(0, 0) = 0.$$

Or, arranging the terms of P_1, Q_1 according to the degree of x, y, it is written in the form

$$\frac{dy}{dx} = \frac{H_m(x, y) + H_{m+1}(x, y) + \dots}{K_n(x, y) + K_{n+1}(x, y) + \dots} \tag{13}$$

where $H_\lambda(x, y)$ and $K_\lambda(x, y)$ are homogeneous integral expressions of λ^{th} degree of x and y.

Now suppose that there exists a characteristic which passes through the origin and has a determinate slope α at that point, then we have

$$\lim_{x \to 0} \frac{y}{x} = \left(\frac{dy}{dx}\right)_{x=0} = \lim_{x \to 0} \frac{P_1(x, y)}{Q_1(x, y)} = \alpha. \tag{14}$$

Using this relation we may find the possible directions.

Case $m > n$. Write (13) in the form

$$\frac{dy}{dx} = \frac{x^{m-n}\{H_m(1, u)+xH_{m+1}(1, u)+\dots\}}{K_n(1, u)+xK_{n+1}(1, u)+\dots}$$

where $u = \frac{y}{x}$; then, by making x converge to zero, by (14) we have

$$\alpha K_n(1, \alpha) = 0$$

if α is finite.

Next, suppose that there exists a characteristic which has the slope ∞ at the origin. Then, writing (13) in the form

$$\frac{dx}{dy} = \frac{K_n(v, 1)+y K_{n+1}(v, 1)+\dots}{y^{m-n}\{H_m(v, 1)+y H_{m+1}(v, 1)+\dots\}}, \quad v = \frac{x}{y},$$

and making y converge to zero, we obtain

$$K_n(0, 1) = 0.$$

Thus, when $m > n$, the possible slopes are

1) $\alpha=0$, 2) roots of $K_n(1, \alpha)=0$ and 3) $\alpha=\infty$ when $K_n(0, 1)=0$.

Case $m = n$. Again put $\frac{y}{x} = u$, then

$$\frac{dy}{dx} = \frac{H_m(1, u)+xH_{m+1}(1, u)+\dots}{K_m(1, u)+xK_{m+1}(1, u)+\dots}$$

whence, by making x converge to zero, we obtain

$$\alpha K_m(1, \alpha) = H_m(1, \alpha)$$

if α is finite.

If the differential equation has any characteristic which touches the y-axis at origin, as in case $m > n$, we have

$$K_n(0, 1) = 0.$$

When $m=n$, the possible slopes at origin are 1) roots of $\alpha K_m(1, \alpha)-H_m(1, \alpha)=0$, 2) $\alpha=\infty$ when $K_m(0, 1)=0$.

Finally, the case $m < n$ may be reduced to the case $m > n$ considering the equation $\frac{dx}{dy} = \frac{Q_1}{P_1}$.

In short the possible directions of the equation (13) are the roots of

$$\sin\theta\, K_s(\cos\theta, \sin\theta)-\cos\theta\, H_s(\cos\theta, \sin\theta) = 0, \tag{15}$$

where s is the lowest degree of the polynomials P_1 and Q_1.

7. Criterions for Existence of Characteristics which have a Definite Possible Direction at a Singular Point.

We have shewn that a characteristic which has a determinate direction at a singular point S may exist only in a possible direction. We inquire conversely whether there will really exist a characteristic in a given possible direction.

To solve this question let ASB be an angle which contains a possible direction α and does not contain any other possible direction. [The case where all directions are possible is treated later].

Consider a sector damain D, bounded by the sides SA, SB of the angle and a circle described with the point S as center and radius R, and suppose that R is chosen so small that in D no singular point is contained and on the side SA no contact point lies.

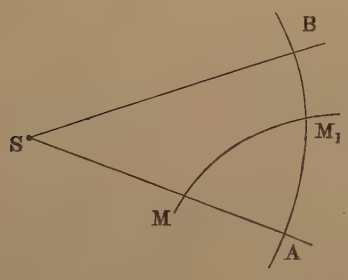


Fig. 10.

Let M be a point on SA and let $C(M)$ be the semi-characteristic through M directed leftwards of SA. Then $C(M)$ has another intersection with the boundary of D. For if $C(M)$ lay entirely in the interior of D, it would be a spiral converging to a limiting cycle (theorem 7). Consequently, by theorem 1, in the interior of D there must exist a singular point. But this is contrary to the supposition. Thus there exists a point M_1 at which $C(M)$ cuts the boundary of D for the first time. We shall use again the term consecutive point for M_1.

As the side SA contains no contact point, the consecutive point M_1 lies on the broken line SBA.

1° Suppose on SA there exists a point M' whose consecutive point is the singular point S. Then $C(M')$ is surely a characteristic which has the slope α at S. For, since $C(M')$ can not make indefinitely many oscillations (theorem 6), it must have a definite direction at S; and in D, as there is only one possible direction α, $C(M')$ must have the slope α at that point. In this case any characteristic $C(N)$ of which N is a point on the segment SM' has the slope α at S (Fig. 11).

2° Suppose, for a sufficiently small R, all the consecutive points of $M's$ lie on the segment AB, then evidently $\lim_{M \to S} C(M)$ is a characteristic in question (Fig. 12).

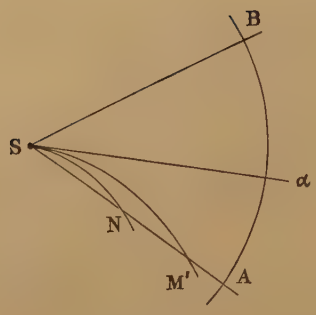


Fig. 11.

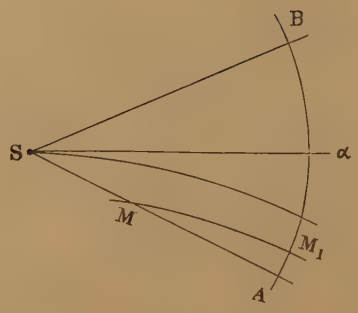


Fig. 12.

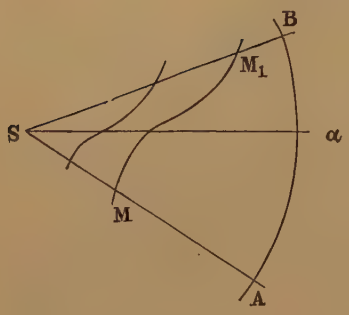


Fig. 13.

3° Suppose, however small R may be taken, on SA there is a point M whose consecutive point M_1 lies on the side SB, then in D there is no characteristic in question (Fig. 13).

Here we remark that the segment of the circle AB and the sides SA, SB may be replaced by suitable curves.

The above criterion may be expressed in a more convenient form as follow:

Criterion 1. Let α be a possible direction at the singular point $S(0, 0)$ of the differential equation (12). Let $D[(\alpha-\delta)x \leqq y \leqq (\alpha+\delta)x, 0 < x \leqq X]$ be a domain where $f(x, y)$ is continuous and in which no other possible direction is contained. For simplicity let the expression $\frac{d\Phi(x)}{dx} - f(x, \Phi(x))$ be denoted by $\chi(x, \Phi(x))$.

Then a) if $\chi(x, (\alpha-\delta)x) < 0$, $\chi(x, (\alpha+\delta)x) > 0$ for all x, $0 < x \leqq X_1$ $(\leqq X)$, there exists at least one characteristic that has the slope α at S; b) if $\chi(x, (\alpha-\delta)x) > 0$, $\chi(x, (\alpha+\delta)x) < 0$ for all x, $0 < x \leqq X_1$, all the characteristics in $[(\alpha-\delta)x \leqq y \leqq (\alpha+\delta)x, 0 < x \leqq X_1]$ pass through S and have the slope α at that point; c) let $\chi(x, (\alpha-\delta)x)$, $\chi(x, (\alpha+\delta)x)$ have the same sign for all x, $0 < x \leqq X_1$, say positive. If there exists a system of curves

$$\gamma(c): \quad y = \Phi(x, c), \quad 0 < c \leqq c_0$$

ending at the lines $y=(\alpha-\delta)x$, $y=(\alpha+\delta)x$, such that the curve converges to S when c converges to zero, and $\chi(x, \Phi(x, c)) > 0$ $(0 < c \leqq c_1)$, in D there is no characteristic through S.

Proof. Consider the domain $D_1[(a-\delta)x \leqq y \leqq (a+\delta)x,\ 0 \leqq x \leqq X_1]$ and let $\varphi(x;\xi,(a-\delta)\xi)$ be the upper semi-characteristic through the point $\xi, (a-\delta)\xi$ $(0 < \xi < X_1)$. By the supposition a), it is clear that the consecutive point of $x=\xi, y=(a-\delta)\xi$, with regard to the boundary of D_1, lies on the side $x=X_1, (a-\delta)X_1 \leqq y \leqq (a+\delta)X_1$; hence the characteristic $\lim_{\xi \to 0} \varphi(x;\xi,(a-\delta)\xi)$ is one in question.

The other two propositions are proved similarly.

Criterion 2. Let $y_1=\psi_1(x), y_2=\psi_2(x)$ be any two different curves through the same singular point $S(x_0,y_0)$ of $\frac{dy}{dx}=f(x,y)$, and suppose $\psi_1(x) < \psi_2(x), x_0 \leqq x \leqq X$. Then, if in $D[\psi_1(x) \leqq y \leqq \psi_2(x), x_0 < x \leqq X]$ $\frac{\partial f}{\partial y}$ is negative, in D there can not exist two characteristics which pass through S.

Proof. Suppose, if possible, there were two such characteristics $y=\varphi_1(x)$ and $y=\varphi_2(x)$ and let

$$\varphi_1 < \varphi_2, \quad x_0 < x \leqq X_1 (\leqq X).$$

Then $$\varphi_2' - \varphi_1' = f(x,\varphi_2) - f(x,\varphi_1) = (\varphi_2-\varphi_1) f_y'(x,\eta),$$

where η is a certain value between φ_1 and φ_2.
Therefore $\varphi_2'-\varphi_1' < 0$ when $x_0 < x \leqq X_1$ and consequently by the condition $\varphi_1(0) = \varphi_2(0) = y_0$ we should have

$$\varphi_2(x) < \varphi_1(x), \quad x_0 < x \leqq X_1$$

which is contradictory.

Criterion 3. Let $y_1 = \psi_1(x)$, $y_2 = \psi_2(x)$ be any two curves through the same singular point $S(x_0, y_0)$ of $\frac{dy}{dx} = f(x,y)$ and let $\psi_1(x) < \psi_2(x)$, $x_0 < x \leqq X$. Then, if in $D[\psi_1(x) \leqq y \leqq \psi_2(x),\ x_0 < x \leqq X] f(x,y)$ is continuous and $\frac{\partial F}{\partial y}$ is negative, where $F(x,y)$ is the concavity function of the characteristics (§ 2), there can not exist two different characteristics which pass through S and have the same slope at S.

Proof. Were there any two such characteristics, let them be called $y=\varphi_1(x)$, $y=\varphi_2(x)$ and suppose

$$\varphi_1(x) < \varphi_2(x), \quad x_0 < x \leqq X_1 (\leqq X).$$

Then $\varphi_2'' - \varphi_1'' = F(x,\varphi_2) - F(x,\varphi_1) = (\varphi_2-\varphi_1) F_y'(x,\eta)$, $\varphi_1 < \eta < \varphi_2$.
Therefore $\varphi_2''(x) < \varphi_1''(x)$, $x_0 < x \leqq X_1$; hence by

$$\varphi_2'(x_0) = \varphi_1'(x_0), \text{ we should have}$$

$\varphi_2'(x) < \varphi_1'(x)$, $x_0 < x \leqq X_1$, and consequently, since $\varphi_1(x_0) = \varphi_2(x_0)$

$= y_0$, the relation $\varphi_2(x) < \varphi_1(x)$, $x_0 < x \leqq X_1$ which is contradictory.

Criterion 4. Let $S(x_0, y_0)$ be a singular point of $\frac{dy}{dx} = f(x, y)$ and let $y=\varphi(x)$ be a characteristic which satisfies $\varphi(x_0)=y_0$, $\varphi'(x_0)=\alpha$. Let $y=\psi(x)$ be a curve of which $\psi(x_0)=y_0$, $\psi'(x_0)=\alpha$ and $\varphi(x) < \psi(x)$, $x_0 < x \leqq X$. Then, if in $D[\varphi \leqq y \leqq \psi,\ x_0 < x \leqq X]f(x,y)$ is continuous and $\chi(x, \psi(x)) \equiv \psi'(x) - f(x, \psi(x)) < 0$, all the characteristics $y=\varphi(x; \xi, \psi(\xi))$, $(x_0 < \xi \leqq X)$, pass through the singular point S and have the slope α at that point.

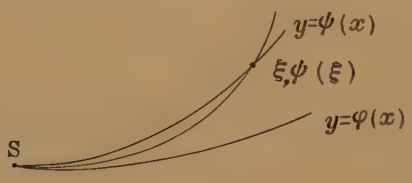


Fig. 14.

Proof. The left semi-characteristic $y=\varphi(x; \xi, \psi(\xi))(x \leqq \xi)$ through $\xi, \psi(\xi)$ lies below the curve and as it can not cut either of the two curves, $y=\varphi(x)$, $y=\psi(x)$ in $(x_0 < x \leqq \xi)$, it must go to the singular point S. From $\varphi(x) \leqq \varphi(x; \xi, \psi(\xi)) \leqq \psi(x)$, $x_0 \leqq x \leqq \xi$ and $\varphi'(x_0)=\alpha$, $\psi'(x_0)=\alpha$ it follows $\varphi'(0; \xi, \varphi(\xi))=\alpha$.

8. Discussion of Particular Possible Directions.

We shall now consider separately each case of possible directions introduced in § 6.

I. $m > n$. $\frac{dy}{dx} = \frac{x^{m-n}\{H_m(1,u) + xH_{m+1}(1,u) + \dots\}}{K_n(1,u) + xK_{n+1}(1,u) + \dots}$, $u = \frac{y}{x}$.

1° $\alpha=0$. When $K_n(1,0) \neq 0$ the equation has at least one integral on each side of the origin, which satisfies the conditions $y(0)=0$, $y'(0)=0$.

Proof. By the conditions $m-n \geqq 1$ and $K_n(1,0) \neq 0$, we may determine positive numbers δ and X such that in $D[-\delta x \leqq y \leqq \delta x,\ 0 < x \leqq X]\ f(x,y)$ is continuous and

$$\chi(x, \delta x) = \delta - f(x, \delta x) > 0,\ \chi(x, -\delta x) = -\delta - f(x, -\delta x) < 0$$

for all values of x, $0 < x \leqq X$. Hence there exists a characteristic in question. The existence of such integral on the left of the origin is proved similarly.

In the exceptional case $K_n(1,0)=0$, the equation may or may not have the characteristic in question as shewn in the example.

Ex. The equation $\frac{dy}{dx} = -\frac{x^3}{y}$ has no integral through the origin.

but $\frac{dy}{dx} = \frac{x^3}{y}$ has two integrals $y = \frac{x^2}{\sqrt{2}}$, $y = -\frac{x^2}{\sqrt{2}}$ subject to the conditions $y(0)=0$, $y'(0)=0$.

2° α is a root of $K_n(1, u)=0$. The differential equation may or may not have an integral subject to the conditions $y(0)=0$, $y'(0)=\alpha$.

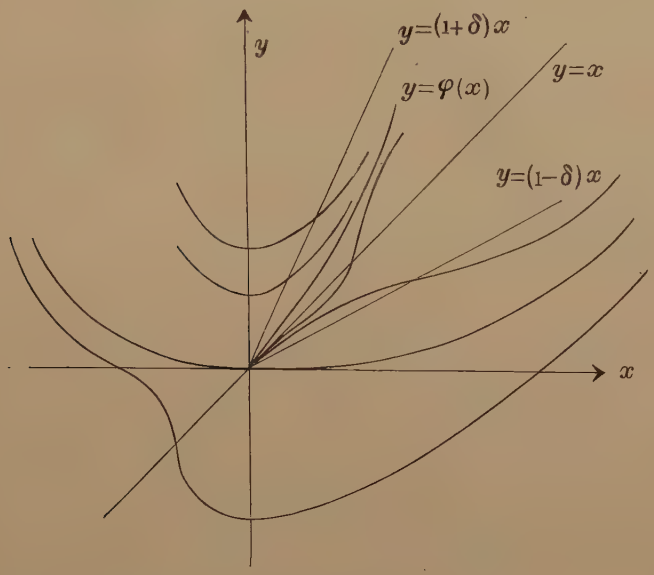


Fig. 15.

Ex. $\frac{dy}{dx} = \frac{x^3}{(y-x)^2}$, (Fig. 15).

Here $f(x,y) = \frac{x^3}{(y-x)^2}$ and it is zero when $x=0, \infty$ when $y=x$ and at any other point it is positive or negative according as x is positive or negative. The possible slopes at the origin are $\alpha=0$, $\alpha=1$, and we consider the latter.

Now $\chi(x, (1+\delta)x) = 1+\delta-\frac{x}{\delta^2}$, $\chi(x,(1-\delta)x)=1-\delta-\frac{x}{\delta^2}$, in which δ is a positive constant less than 1, are positive when x is sufficiently small. Also we have

$$\chi(x,x)=-\infty \text{ when } x>0 \text{ and } \chi(x,x)=+\infty \text{ when } x<0.$$

As $\chi(x,(1+\delta)x)$ and $\chi(x,x)$ have different signs, in $D[x \leqq y \leqq (1+\delta)x, 0<x\leqq X]$ X being a sufficiently small positive value, by criterion 1 a) the equation has an integral $y=\varphi(x)$ which satisfies the conditions $\varphi(0)=0$, $\varphi'(0)=1$.

Since $\chi(x,(1-\delta)x)>0$ when $0<x\leqq X_1$, where X_1 is a positive value, all the characteristics in $[(1-\delta)x\leqq y\leqq\varphi(x),\ 0<x\leqq X_1]$ pass through the origin having the slope $+1$ at that point.

When $x<0$, $f(x,y)$ is negative and there is no characteristic which passes through the origin in the third quadrant.

3° $\qquad a=\infty,\ K_n(0,1)=0.$

Ex. $\dfrac{dy}{dx}=\dfrac{y^3}{x^2-y^3}$. The upper semi-characteristics $\varphi(x;\xi,\xi^{\frac{2}{3}})$, $\xi<0$, lie between the positive y-axis and the curve $y=x^{\frac{2}{3}}$, $x<0$, hence $\lim_{\xi\to 0}\varphi(x;\xi,\xi^{\frac{2}{3}})$ is a characteristic which touches the positive y-axis at the origin. Below the x-axis there is no characteristic which touches the y-axis at the origin (Fig. 16).

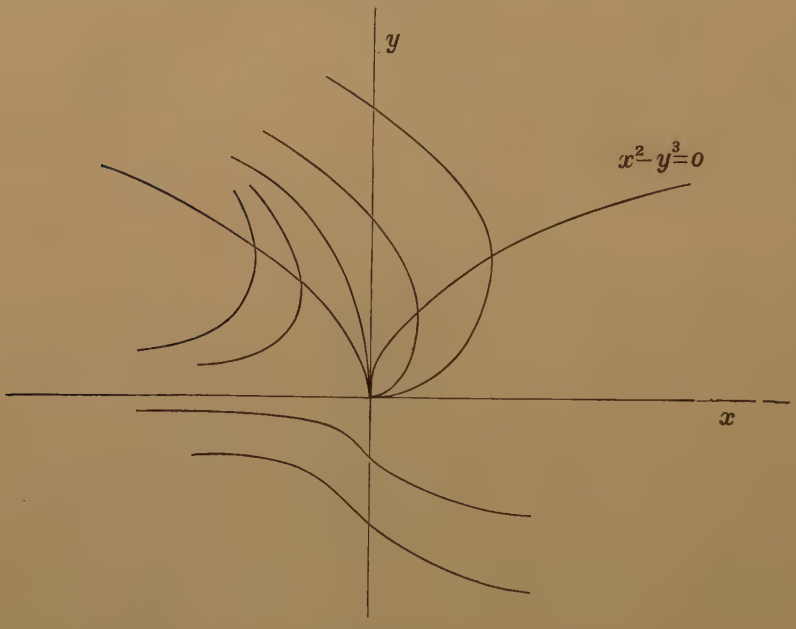


Fig. 16.

II. $m=n$. $\qquad \dfrac{dy}{dx}=\dfrac{H_m(x,y)+H_{m+1}(x,y)+\cdots}{K_m(x,y)+H_{m+1}(x,y)+\cdots}$.

The possible directions of this equation are 1) roots of

$(15)_a \qquad \pi(u)\equiv uK_m(1,u)-H_m(1,u)=0$

and 2) $a=\infty$ if $K_m(0,1)=0$.

In case 2), if we interchange the variables x and y, it will be re-

duced to case 1) corresponding to the slope $a=0$. Hence we have only to consider the existence of integrals of the form

$$y = x(a+\varepsilon(x))$$

where a is a root of $(15)_a$ and $\varepsilon(x)$ is a variable which vanishes with x.

1° a is a common root of $u K_m(1,u)=0$ and $H_m(1,u)=0$.

Ex. $\frac{dy}{dx} = \frac{y-x^2+2x^3}{y}$. $a=0$ is a common root of $H_1(1,u)=u=0$ and $u K_1(1,u)=u^2=0$ and there is the integral $y=x^2$ of the required form.

$\frac{dy}{dx} = \frac{y^2+x^4}{y^2}$ has no integral which satisfies $y(0)=0$ and $y'(0)=0$, since $\frac{dy}{dx}$ is always greater than 1.

2° a is a root of $\pi(u)=0$ but not of $K_m(1,u)=0$. Put $y=(a+y_1)x$, then

$$x\frac{dy_1}{dx}+a+y_1 = \frac{H_m(1,a+y_1)+x H_{m+1}(1,a+y_1)+\dots}{K_m(1,a+y_1)+x K_{m+1}(1,a+y_1)+\dots}$$

or since $K_m(1,a) \neq 0$, $\pi(a) = a K_m(1,a) - H_m(1,a) = 0$, it may be written in the form

$$x\frac{dy_1}{dx} = \frac{ax+by_1+\dots}{1+Ax+By_1+\dots} \equiv R(x,y_1) \tag{16}$$

where $R(x,y_1)$ is a rational function. The differential equation of form (16) will be considered in subsequent article.

3° Case where $\pi(u)=0$ is an identity. In this case all directions are possible.

Put $y = xy_1$, then, remembering $y_1 K_m(1,y_1) - H_m(1,y_1) = 0$, the differential equation will be transformed into

$$\frac{dy_1}{dx} = \frac{L_m(y_1)+x L_{m+1}(y_1)+\dots}{K_m(1,y_1)+xK_{m+1}(1,y_1)+\dots} \equiv R(x,y_1)$$

where $L_i(y_1)$ are polynomials of y_1 and $R(x,y_1)$ is a rational function.

Let c be any value for which $K_m(1,c) \neq 0$, then R and $\frac{\partial R}{\partial y_1}$ are continuous functions of x and y_1 in the neighbourhood of $x=0, y_1=c$. Therefore, by Cauchy's theorem, there exists one integral of $\frac{dy_1}{dx} = R(x,y_1)$, which assumes the value c at $x=0$. The original equation has therefore the integral of the form

$$y = \{c+\varepsilon(x,c)\}x$$

where $\lim_{x\to 0} \varepsilon = 0$ and c is an arbitrary constant for which $K_m(1,c) \neq 0$.

To know if the given differential equation will have an integral which has the direction of the y-axis at origin, put $\frac{x}{y}=v$ and consider y as the independent variable. Then in the same way we may prove the existence of the integrals

$$x = \{c' + \varepsilon_1(y, c')\}\, y$$

where $\lim_{y\to 0} \varepsilon_1 = 0$ and c' is an arbitrary constant for which $K_m(c', 1) \neq 0$. The condition $y'_{x=0} = \infty$ corresponds to $c' = 0$.

Thus, when $\pi(u)=0$ is an identity, there exists one integral on each side of the origin, which has an arbitrarily given slope c at the origin, provided that c is not the slope of the lines $\frac{K_m(x,y)}{x} = 0$.

In the exceptional directions there may or may not exist integrals

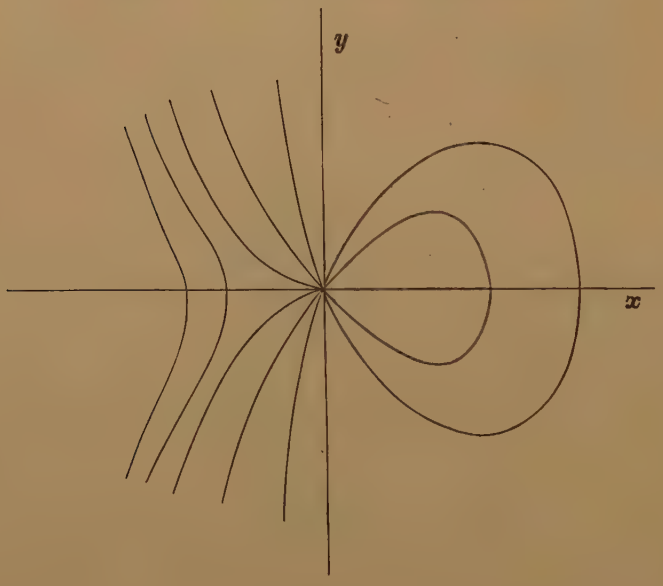


Fig. 17.

as illustrated by the example.

Ex. $$\frac{dy}{dx} = \frac{y^2 - x^3}{xy} \qquad \text{(Fig. 17)}.$$

The exceptional direction is $\theta = 0$. On the right of the origin there is no integral which satisfies $y(0)=0$, $y'(0)=0$; but on the left there are two such integrals $y = \sqrt{2}\,(-x)^{\frac{3}{2}}$ and $x = -\sqrt{2}\,(-x)^{\frac{3}{2}}$. The general integral of the equation is $y^2 = cx^2 - 2x^3$.

9. Form of Characteristics in the Neighbourhood of a Singular Point.

To know the nature of characteristics near a singular point S, consider a closed curve γ which contains only the singular point S in the interior. Such a curve is called a *cycle about* S. The simplest cycle about S is a circle of the center S, which contains no other singular point. We shall call it a circle about S. We here consider the form of characteristics which lie in a cycle K about S.

With respct to K the characteristics are divided into five types:

i) Both sides may end at K (characteristic of *parabolic* form);

ii) One side may end at K and the other side at the singular point S having a definite direction at that point (characteristic of *ray* form);

iii) Both sides may end at S (characteristic of *loop* form);

iv) The characteristic may be a spiral;

v) It may be a cycle about S.

Of course these types are not essential of the characteristics but depend upon the cycle of reference, e.g. a characteristic of ray form with regard to one cycle, may be a loop with regard to a larger cycle.

We consider first a singular point where no characteristic passes, which has a definite direction at that point.

Let l be a straight line through S, then as this line is not a characteristic we may determine a segment SM which contains no contact point. Let K be a cycle about S which passes through M and has no other intersection with the segment SM.

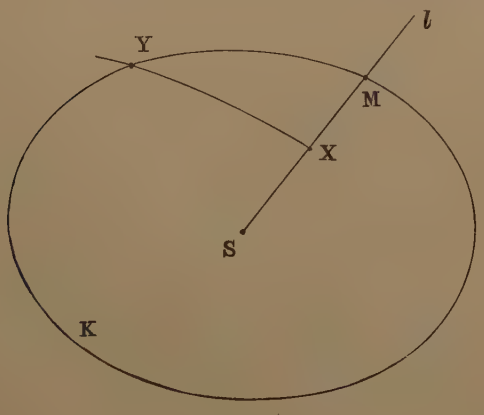


Fig. 18

Let $C(X)$ be the left semi-charateristic through a point X on l_{SM} and consider the curve K_1 which consists of l_{SM} and the cycle K. With respect to K_1, the point X has the consecutive point Y which is different from S. Further we know that the consecutive points $Y's$ corresponding to $X's$ $(S < X \leqq M)$ can not all lie on K. For were this the

case, denoting $\lim_{X \to S} Y$ by Y_S, the characteristic $C(Y_S)$ would be a ray contrary to supposition. Hence on l_{SM} there must be a point, call it X_1, whose consecutive point Y_1 lies on l_{SM}.

When $X_1 = Y_1$ the characteristic $C(X_1)$ is a cycle about S, and all the characteristics within it must be cycles or spirals.

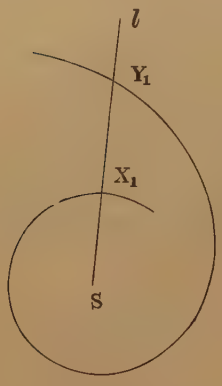


Fig. 19.

When X_1 is different from Y_1, Y_1 is a consecutive point of the second kind, and the segment $C(X_1Y_1)$ of the characteristic $C(X_1)$, intercepted between X_1, Y_1 with the segment $l_{X_1Y_1}$, will form a cycle K_2 about S. Within K_2 there is no characteristic of ray or loop or parabolic form and all the characteristics must be cycles or spirals.

Theorem 8. If through a singular point S there is not any characteristic which has a definite direction at S, then a neighbourhood of S can be constructed such that all the characteristics within it are spirals or cycles.

Suppose the consecutive point Y_1 of X_1 lies between S and X_1. Then $C(X_1)$ is a spiral and it will converge to the point S or to a limiting cycle about S. In the first case all the characteristics in the cycle K_2 are spirals converging to the singular point S. In the second case the limiting cycle C_L and the cycle K_2 will form a ring domain R; and all the characteristics in R will be spirals converging to the same limiting cycle C_L.

If all the characteristics in a neighbourhood of a singular point S are cycles, the point S is called a *center*. On the contrary, if in any cycle about S there is a characteristic of spiral form, the point S is called a *focus*.

Using the terms we may say: if through a singular point S no characteristic which has a determinate direction at that point passes, the point S is a focus or a center.

Remark 1. If at a singular point S no possible direction exists, S is a focus or a center.

Remarks 2. The characteristic angle belonging to a focus or to a center is 2π.

Remark 3.[1] Prof. Poincaré has shewn that if in a cycle K about S there exists an infinite number of cyclic characteristics which do not have S as a limiting point, then all the characteristics in the neighbourhood of S are cycles and the point S is a center.

[1] J. Liouville pp. 256–257 (1882).

10. Form of Characteristics in the Neighbourhood of a Singular Point which is neither a Focus nor a Center.

Let S be a singular point which is neither a focus nor a center, then through S at least one characteristic passes which has a definite direction at S. Consider a circle K about S which cuts the characteristic. Then, since the circle K is an algebraic curve and is not a characteristic, it contains a finite number of contact points. Of these contact points if there are some at which the semi-characteristics have parabolic form these semi-characteristics will divide the domain bounded by K into a finite number of parts.

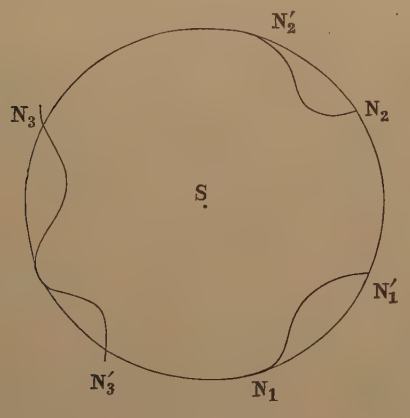


Fig. 20.

One of the subdomains contains S in the interior, call it D_1. It will generally be bounded by a finite number of segments of the circle K, $KN_1'N_2$, $KN_2'N_3$, ... $KN_n'N_1$ and a finite number of characteristics $C(N_1N_1')$, $C(N_2N_2')$ ... $C(N_nN_n')$ as shewn in Fig. 20. The boundary B_1 of D_1 may also contain a finite number of contact points M_1, M_2, ... M_m not belonging to the segments $\{(NN')\}$, and we shall obtain an aggregate $\{M, (NN')\}$ of contact points and segments of characteristics.

Let $C_1(M)$, $C_2(M)$ be the two semi-characteristics at a contact point M and let $C_1(N)$, $C_2(N')$ be the semi-characteristics at N and N', which are the extensions of the characteristic $C(N, N')$. The aggregate $\{M, (NN')\}$ is divided into the following three classes:

a) Those points or segments at which both the semi-characteristics C_1 and C_2 lie out of D_1 (Fig. 21);

b) Those at which both lie in D_1 (Fig. 22);

c) Those at which one lies out of D_1 and the other lies in D_1 (Fig. 23).

It is to be noticed that when the semi-characteristic at a point M or N enters into the interior of D_1, it must be a ray, as it can not be parabolic.

Let $C(N, N')$ be a segment belonging to a) and let L be the ad-

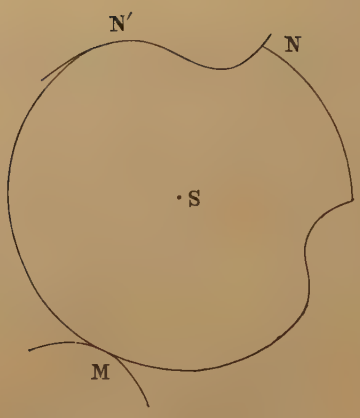


Fig. 21.

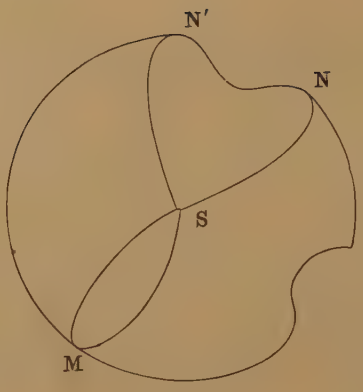


Fig. 22.

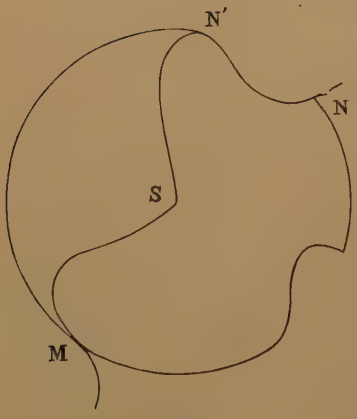


Fig. 23.

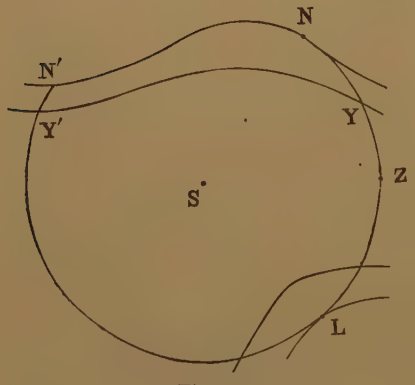


Fig. 24.

jascent point of N, which is different from N' and belongs to the aggregate $\{M, N, N'\}$. L is a contact point or an endpoint of another characteristic $C(\bar{N}, \bar{N}')$, and in $K(L, N)$ there is no contact point (Fig 24).

Let Y be a point of $K(L, N)$, then, remembering that the characteristic through a point P moves continuously with the point P, we see that when Y is sufficiently near to N the characteristic $C(Y)$ is parabolic and separates the characteristic $C(N, N')$ from the singular point S. The point Y has therefore its consecutive point Y_1 with respect to the boundary B_1.

Now divide all the points on $K(L, N)$ into two groups A, B by the following rule: Assume the order LNN' and let Z be any point on $K(L, N)$. 1° If all the characteristics $C(Y)$, $Z < Y < N$, be parabolic and separate $C(N, N')$ from S, we put the point Z into group B. 2° If the point Z does not satisfy the condition 1°, we put it into group A.

Then the partition A, B will determine a point E, which may coincide with L when the characteristic $C(L)$ is a ray.

Let $\lim_{Y \to E+0} C(Y, Y') = C(E, E')$, then $C(E, E')$ is a characteristic which passes through the point S. For, were $C(E, E')$ a characteristic of parabolic form, suppose it lay on the same side of S with $C(N, N')$, in $K(L, E)$ there would be a point Z through which a characteristic passes that has the same property as $C(E, E')$ and the point $Z(<E)$ would belong to group B. This is impossible. Similarly $C(E, E')$ can not be a parabolic characteristic which lies on the opposite side of S with $C(N, N')$. Thus the curve $C(E, E')$ must pass through S, and it will generally consist of two rays and a number of loops.

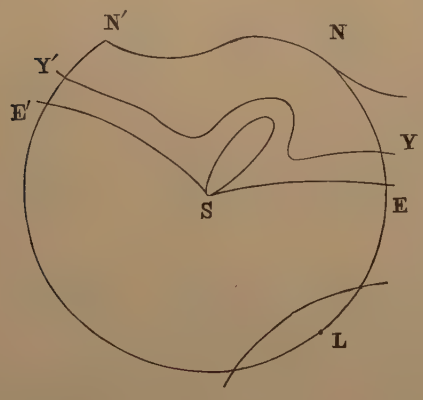


Fig. 25.

It is generally shewn that the number of loops which have the common vertex S and of which any lies out of the others must be finite. For, supposing there be l such loops, draw a circle about S, which cuts all the loops, then, in each segment of the circle intercepted by one loop there exists at least one contact point and in the circle there are at least l contact points. Now, by theorem IV, when the degree of $f(x, y) = \frac{P}{Q}$ is m, any circle can not have more than $2(m+1)$ contact points. This shews that through the point S there can not be more than $2(m+1)$ loops of the said property.

Thus, when $C(NN')$ be a segment of a) class, there exist two ray characteristics SE, SE' and finite loops which with the segment $ENN'E'$ (of B_1) form a domain where all the characteristics are parabolic (Fig. 25).

The same may be said of a contact point of class a) i.e. when M is a point of a) there exist two ray characteristics SF, SF' and finite loops such that in the domain, bounded by the curves and the segment FMF' of B_1, all characteristics are parabolic. We shall call these domains *parabolic domains*.

We consider now the elements of class b).

Let a point M or an arc $C(NN')$ be an element of class

b). Then two ray characteristics $C_1(M)$, $C_2(M)$ or three characteristics $C_1(N)$, $C_2(N')$ and $C(NN')$ will form a loop, in the interior of which all characteristics are loops. The same may be said of any loop which forms the boundary of a parabolic domain. These domains are called *loop domains.*

From D_1 when all parabolic and loop domains are removed, there will remain a finite number of sector domains in which the characteristics are rays. We call these *ray-domains.*

Theorem 9. When S is a singular point which is neither a focus nor a center, there exists a neighbourhood D_1 of S and a finite number of characteristics of ray and loop forms, which divide D_1 into subdomains of the three kinds: parabolic, ray and loop.

A singular point is called a *node* when all the characteristics in a neighbourhood of the point are rays.

From theorems 8, 9 it follows:

Cor. If there exists a sequence of cycles about a singular point S, which converges to S and has no contact point, then the singular point S is a focus or a node.

We can now determine what form a limiting cycle may have.

Let C_L be a limiting cycle of a spiral C. If C_L does not pass through any singular point it is a regular closed curve.

When it passes through singular points it will consist of characteristics whose extremities are singular points. Let S_1 be a singular point on C_L and $C_1(S_1)$ be a branch of C_L through S_1. If $C_1(S_1)$ is not a loop it is a ray which goes to another singular point. That the number of characteristics belonging to C_L, which pass through the point S_1, is finite, may be proved as in theorem 9.

It will be further shewn that the number of branches of C_L at S_1 is even. If there were an odd number of branches of C_L through S_1, there must one, say $C_i(S_1)$, on both sides of which the spiral C would lie. Let M_1MM_2 be a

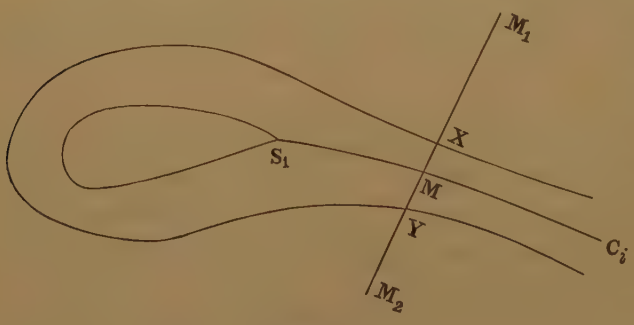


Fig. 26.

segment of a straight line which contains a point M on $C_i(S)$ and does not contain any contact point. Then on each segment MM_1, MM_2 there would be indefinitely many points on C and we might determine a point X on M_1M, which would be common to C, such that the consecutive point Y with respect to the segment M_1MM_2 would be on MM_2 and be of the first kind. This would bring a contact point on M_1M_2, which is contradictory.

Hence we may conclude: any limiting cycle may consist of a finite number of loops and a finite number of ray characteristics which connect different singular points, and at each singular point meet an even number of branches (Fig. 9).

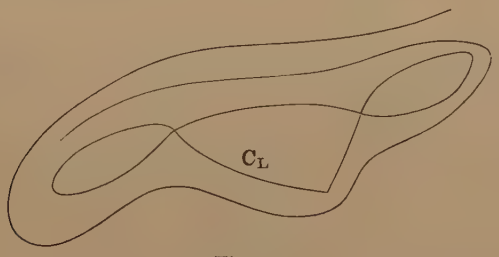


Fig. 9.

11. Form of Characteristics in an Infinite Distance.

If there be any characteristic $y=\varphi(x)$ of a differential equation

$$\frac{dy}{dx}=f(x,y)=\frac{P(x,y)}{Q(x,y)}$$

which has a definite direction at infinite distance, then the three limiting values

$$\lim_{x\to\infty}\frac{dy}{dx},\quad \lim_{x\to\infty}\frac{y}{x}\quad\text{and}\quad \lim_{x\to\infty}f(x,y)$$

will have the same value. Using these conditions we can determine the possible directions of characteristics at infinite distance.

For the purpose, arrange the terms of $P(x,y)$ and $Q(x,y)$ in descending powers of x and y, thus

$$\frac{dy}{dx}=\frac{H_m(x,y)+H_{m-1}(x,y)+\dots}{K_n(x,y)+K_{n-1}(x,y)+\dots},$$

where $H_i(x,y)$, $K_i(x,y)$ are homogeneous expressions of i^{th} degree. Treating this in a similar manner as we did a singular point, we find that the possible directions at infinite distance are:

i) $m > n$, $\cos\theta\, H_m(\cos\theta, \sin\theta) = 0$;
ii) $m = n$, $\sin\theta\, K_m(\cos\theta, \sin\theta) - \cos\theta\, H_m(\cos\theta, \sin\theta) = 0$;
iii) $m < n$, $\sin\theta\, K_n(\cos\theta, \sin\theta) = 0$.

To examine the existence of characteristics in a possible slope α we use the projective transformation

$$x = \frac{1}{X}, \quad y = \frac{Y}{X} \tag{17}$$

if $\alpha \neq \infty$ (i.e. $\theta \neq \frac{\pi}{2}$). Then

$$\frac{dY}{dX} = \frac{Y - f\left(\frac{1}{X}, \frac{Y}{X}\right)}{X} \tag{18}$$

$$= \frac{YX^{m-n}\{K_n(1,Y) + XK_{n-1}(1,Y) + \ldots\} - \{H_m(1,Y) + XH_{m-1}(1,Y) + \ldots\}}{X^{m-n+1}\{K_n(1,Y) + XK_{n-1}(1,Y) + \ldots\}}$$

and we have to consider the characteristics at $X = 0$, $Y = \alpha$ of the new equation. If (18) has a characteristic of the form

$$Y = \alpha + X(\beta + \varepsilon(X)), \quad |X| < \delta$$

where $\lim_{X \to 0} \varepsilon(X) = 0$, then the original equation has a characteristic of the form

$$y = \alpha x + \beta + \varepsilon\left(\frac{1}{x}\right), \quad |x| > \frac{1}{\delta}.$$

When $\alpha = \infty$ or α is a possible slope which is not zero, put

$$X = \frac{1}{y}, \quad Y = \frac{x}{y} \tag{19}$$

then the equation (2) is transformed into

$$\frac{dY}{dX} = \frac{f\left(\frac{Y}{X}, \frac{1}{X}\right) Y - 1}{Xf\left(\frac{Y}{X}, \frac{1}{X}\right)} \tag{20}$$

$$= \frac{Y\{H_m(Y,1) + XH_{m-1}(Y,1) + \ldots\} - X^{m-n}\{K_n(Y,1) + XK_{n-1}(Y,1) + \ldots\}}{X\{H_m(Y,1) + XH_{m-1}(Y,1) + \ldots\}}$$

and we have to consider the characteristics through the point $X = 0$, $Y = \frac{1}{\alpha}$, when α is finite, or the new origin, when α is infinite.

By the transformation (17) or (19) the infinity line in the xy-plane is transformed into the Y-axis of the new plane; and by the equation

(18) or (20) we see that, with the exception of the case where $m=n$ and $\sin\theta K_m(\cos\theta, \sin\theta)-\cos\theta H_m(\cos\theta, \sin\theta)=0$ is an identity, the Y-axis ($X=0$) is a characteristic and any possible direction at infinite distance is transformed into a singular point which lies on the Y-axis of the new plane.

12. Singular Points of the First Order. Focus and Center.

Suppose a singular point S of a differential equation (2) is taken for the origin, then we have

$$\frac{dy}{dx}=\frac{ax+by+\dots}{Ax+By+\dots}. \tag{21}$$

If in this equation $D\equiv aB-Ab\neq 0$, the origin is called a *singular point of the first order*. In the following we shall consider the characteristics of (21) in the neighbourhood of the origin.

The equation for possible slopes of (21) is

$$Bt^2+(A-b)t-a=0 \tag{22}$$

and there are two real different, or real coincident, or no real possible slopes, according as the discriminant

$$E\equiv(A-b)^2+4aB$$

is positive, zero or negative. We consider these cases separately.

When E is negative, the origin is a focus or a center as there is no possible direction at that point (§ 9, Remark 1).

To find the form of characteristics near the origin more exactly, put

$$y=r\sin\theta, \quad x=r\cos\theta,$$

then (21) becomes

$$\frac{r'}{r}=\frac{A\cos^2\theta+(B+a)\cos\theta\sin\theta+b\sin^2\theta+r(\quad)}{a\cos^2\theta+(b-A)\cos\theta\sin\theta-B\sin^2\theta+r(\quad)}, \tag{23}$$

where the expresssions in the brackets are polynomials of r, $\cos\theta$ and $\sin\theta$, and, by the form of the equation, we know that, for sufficiently small r, $\frac{dr}{d\theta}$ is continuous of r and θ.

Neglecting higer infinitesimals the last equation becomes

$$\frac{R'}{R}=\frac{A\cos^2\theta+(B+a)\cos\theta\sin\theta+b\sin^2\theta}{a\cos^2\theta+(b-A)\cos\theta\sin\theta-B\sin^2\theta};$$

and, integrating this, we obtain

$$R = C\frac{e^{M \operatorname{arctg} G}}{\sqrt{a\cos^2\theta + (b-A)\cos\theta\sin\theta - B\sin^2\theta}}$$

where $$M = \frac{b+A}{\sqrt{-E}}, \quad G = \frac{-2B\tan\theta + b - A}{\sqrt{-E}}$$

and C is an arbitrary constant. The integrals are spirals if $M \neq 0$, since

$$R(\theta + 2\pi) = e^{\sigma 2M\pi} R(\theta)$$

where σ is 1 or -1 according as B is negative or positive.

As the integral r differs from R by an infinitesimal, we may say that the characteristics of (23) in the neighbourhood of origin are spirals converging to the origin when $E < 0$ and $A+b \neq 0$.

When $A+b=0$ the integrals R are cycles since $R(\theta+2\pi)=R(\theta)$. It can not be inferred that the same holds for the integrals r. We may so far only conclude that the origin may be a center when $E<0$ and $A+b = 0$.[1]

13. Singular Points of the First Order which is neither a Focus nor a Center.

Though equation (21), of which $E=(A-b)^2+4aB$ is not negative may be treated directly, it is more convenient to consider the reduced equation.

Put

(24) $$x = \xi + \eta, \quad y = \alpha\xi + \beta\eta$$

then

(25) $$\frac{d\eta}{d\xi} = \frac{a_1\xi + b_1\eta + \dots}{A_1\xi + B_1\eta + \dots}$$

where

(26) $$\left\{\begin{array}{ll} a_1 = -\{B\alpha^2 + (A-b)\alpha - a\}, & b_1 = a + b\beta - A\alpha - B\alpha\beta, \\ A_1 = -a + A\beta - b\alpha + B\alpha\beta, & B_1 = B\beta^2 + (A-b)\beta - a. \end{array}\right\}$$

When $D > 0$ the two roots of (22) are real and different, call them t_1, t_2, and in the equations of transformation put $\alpha = t_1, \beta = t_2$, then we have

$$a_1 = 0, \quad b_1 = (t_2 - t_1)(Bt_2 + A), \quad A_1 = (t_2 - t_1)(Bt_1 + A), \quad B_1 = 0$$

and (25) becomes

[1] A rigorous criterion to distinguish center and focus was given by Prof. Poincaré in J. Liouville, pp. 173–196 (1885).

(27) $$\frac{d\eta}{d\xi} = \frac{\lambda\eta + \dots}{\xi + \dots}$$

where $\lambda = \dfrac{Bt_1 + A}{Bt_2 + A}$. The value of λ is the ratio of the two roots of the equation

$$\rho^2 - (A+b)\rho + Ab - aB = 0;$$

and it is positive or negative according as D is negative or positive.

When $D=0$ the two roots of (22) are equal, say t_1, and put t_1 for β in (23), then

$$a_1 = -B(t_1 - \alpha)^2, \quad A_1 = b_1 = -Bt_1^2 + t_1(b + B\alpha) - b\alpha, \quad B_1 = 0$$

and (25) becomes

(28) $$\frac{d\eta}{d\xi} = \frac{\eta + \lambda\xi + \dots}{\xi + \dots}$$

where $$\lambda = \frac{B(t_1 - \alpha)^2}{Bt_1^2 - t_1(b - B\alpha) + b\alpha}, \quad t_1 = \frac{A-b}{2B}$$

and α is a value which is not equal to t_1.

As the transformation (24) is linear homogeneous, to a cycle γ about the origin of xy-plane, a cycle γ_1 about the origin of $\xi\eta$-plane corresponds and to a characteristic C in the old, a characteristic C_1 in the new plane. Also it is evident that the type of C with regard to γ is the same as that of C_1 with regard to γ_1. Thus to find the form of characteristics of (21) we have to investigate the charactcristics of the transformed equation. It is also to be noticed that in the transformed equation (27) the possible directions are $\tau_1 = 0$, $\tau_2 = \infty$ and that these correspond to t_1, t_2 of the original equation.

1° $$\frac{d\eta}{d\xi} = \frac{\lambda\eta + \dots}{\xi + \dots} = f(\xi, \eta), \quad \lambda < 0.$$

To apply criterion 1 § 7, take two lines $y = lx$ $y = -lx$ where l is a positive number. Since $\lim_{x \to 0} f(x, ux) = \lambda u$, we may determine a positive value X such that, in $D[-lx \leqq y \leqq +lx, 0 < x \leqq X]$ $f(x,y)$ is continuous and

$$f(x, -lx) > -l, \quad f(x, lx) < l \quad (0 < x \leqq X).$$

Hence in D there exists a characteristic $y = \varphi_0(x)$ which satisfies $\varphi_0(0) = 0$, $\varphi_0'(0) = 0$. Besides $y = \varphi_0(x)$, in D there is no characteristic which passes through the origin, as

$$\frac{\partial f}{\partial y} = \frac{\lambda x + \dots}{(x+\dots)^2}$$

is negative in $[-lx \leqq y \leqq +lx, 0 < x \leqq X_1]$, when X_1 is sufficiently small.

Similarly, it is shewn that in the direction $\theta=\pi$ there exists a single integral.

To find the existence of characteristics in the directions $\theta=\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$, write the equation in the form

$$\frac{d\xi}{d\eta} = \frac{\frac{1}{\lambda}\xi + \dots}{\eta + \dots}, \quad \frac{1}{\lambda} < 0,$$

and, considering η as the independent variable, we may conclude that there exists a single characteristic in each direction of $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$.

Thus the equation (27) has four ray characteristics in the directions $\theta=0$, $\theta=\frac{\pi}{2}$, $\theta=\pi$ and $\theta=\frac{3\pi}{2}$; and all the other characteristics near the origin are parabolic.

A singular point of this sort is called a *col*.[1]

Remark. At a *col* the characteristic angle is -2π.

2° $$\frac{d\eta}{d\xi} = \frac{\lambda\eta + \dots}{\xi + \dots} \equiv f(\xi, \eta), \quad 0 < \lambda.$$

We treat first the case $0 < \lambda < 1$. To find the existence of characteristics in the direction $\theta=0$, again apply criterion 1 § 7. We have $\lim_{x \to 0} f(x, ux) = \lambda u$ and there exists a positive number X such that in $[lx \leqq y \leqq +lx, 0 < x \leqq X]$, $f(x, y)$ is continuous and

$$-l - f(x, -lx) < 0, \quad l - f(x, lx) > 0 \quad (0 < x \leqq X)$$

where l is a positive number. This shews that in the direction $\theta=0$ there exists a characteristic. Now by calculation we find

$$\frac{\partial}{\partial \eta}\left(\frac{d^2\eta}{d\xi^2}\right) = \frac{\lambda(\lambda-1)\xi^2 + \dots}{(\xi+\dots)^4} \quad (0 < \lambda < 1)$$

and, as this value is negative in $[-lx \leqq y \leqq +lx, 0 < x \leqq X_1]$ for

[1] As a proper definition is not found, following Prof. Poincaré, the french term 'col' is adopted.

sufficiently small positive X_1, in the direction $\theta=0$ there is not any other characteristic. Similarly, in the direction $\theta=\pi$, there exists only one characteristic.

Next consider a circle of radius ρ which has the origin as center. The contact points of the circle are given by

$$x(x+\ldots)+y(\lambda y+\ldots)=0, \quad x^2+y^2=\rho^2$$

and it is clear that when ρ is sufficiently small the simultaneous equations have no solution. The origin is therefore a node. As the differential equation has only the remaining possible directions $\theta=\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$, the characteristics in the neighbourhood of the origin must have these directions at the origin.

Case $\lambda=1$. In this case the equation for possible directions becomes an identity; and all directions at the origin are possible. Also there is no exceptional direction. The origin is therefore a node; and in each direction at the origin there is only one characteristic.

Case $\lambda > 1$. Writing the equation in the form

$$\frac{d\xi}{d\eta}=\frac{\frac{1}{\lambda}\xi+\ldots}{\eta+\ldots} \qquad 0<\frac{1}{\lambda}<1,$$

and considering η as the independent variable, it reduces to the case $0<\lambda<1$.

3° $$\frac{d\eta}{d\xi}=\frac{\eta+\lambda\xi+\ldots}{\xi+\ldots}, \quad \lambda\neq 0.$$

In this case $\alpha=\infty$ is a double root of the equation (22). By criterion 1 § 7, we see that all characteristics in $[-ly\leqq x\leqq ly,\ 0<y\leqq Y]$, where l and Y are sufficiently small positive values, touch the positive y-axis, and those in $[ly\leqq x\leqq -ly,\ -Y\leqq y<0]$ touch the negative y-axis at the origin. Further, the ellipse $x=\rho\cos\theta$, $y=b\rho\sin\theta$ has no contact point when $b>|2\lambda|$, for sufficiently small ρ. The origin is therefore a node. As there are only two possible directions $\theta=\frac{\pi}{2}, \theta=\frac{3\pi}{2}$, all the characteristics in the neighbourhood of the origin must have these directions at the origin.

The finding concerning characteristics in the neighbourhood of a singular point of the first order may be summed up into the following:

Theorem 10. In the equation

$$\frac{dy}{dx} = \frac{ax+by+\dots}{Ax+By+\dots}, \quad D \equiv \begin{vmatrix} a & b \\ A & B \end{vmatrix} \neq 0$$

i) When $E \equiv (A-b)^2+4aB < 0$, *the origin is generally a focus. It may be a center only when the additional condition* $A+b=0$ *is satisfied;*

ii) When $E>0$ *the origin is a col* (*Fig.* 27) *or a node according as* D *is positive or negative; and specially a) when* $E>0$, $D<0$, *denoting by* t_2 *that one of the possible directions, for which the absolute value of* $Bt+A$ *is greater than that at the other root* t_1, *there are indefinitely many characteristics that have the slope* t_2 *on both sides of the origin, and there is a single characteristic on each side of the origin, that has the slope* t_1 (*Fig.* 28); *b) when* $a=0, B=0$, $A=b$ *on each direction at the origin there is a single characteristic;*

iii) When $E=0$ *the origin is a node and all characteristics have the same slope* $\frac{A-b}{2B}$ *at the origin* (*Fig.* 29).

Equation (16), given in § 8

$$x\frac{dy}{dx} = \frac{ax+by+\dots}{1+A_1x+B_1y+\dots}, \quad b \neq 0$$

is a special case of the above. The possible directions at the origin are $\frac{a}{1-b}$ and ∞; and the origin is a *col* or a node according as b is negative or positive.

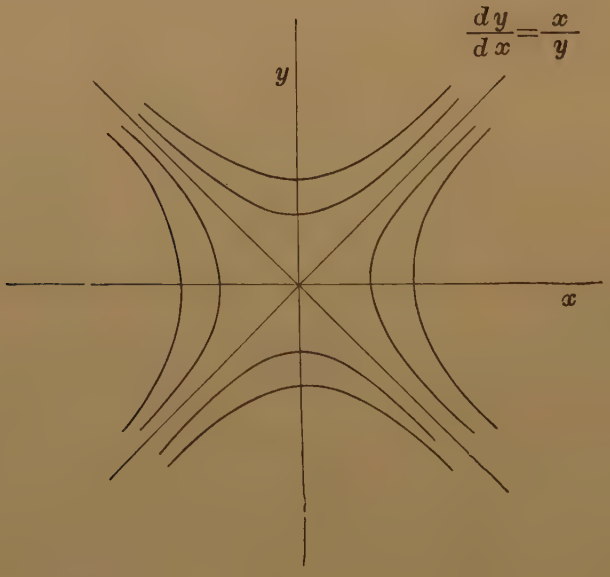


Fig. 27.

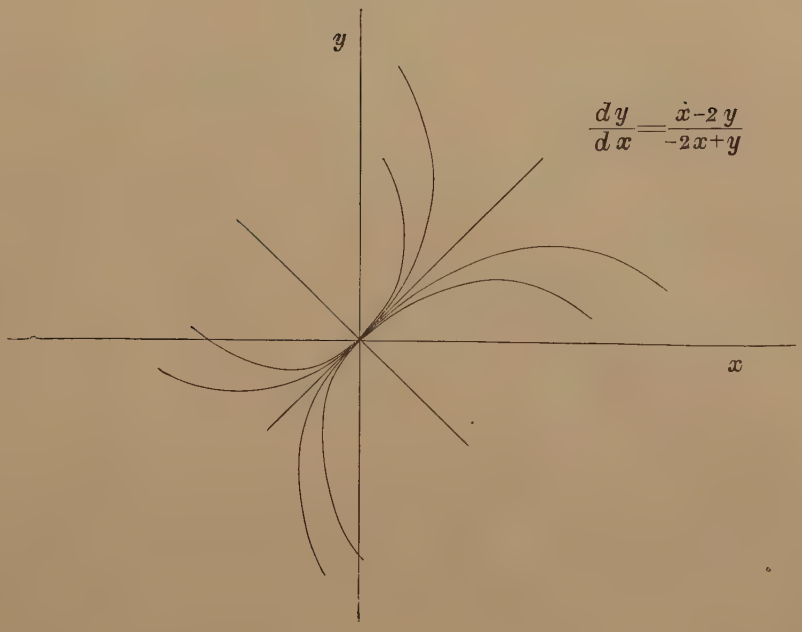


Fig. 28.

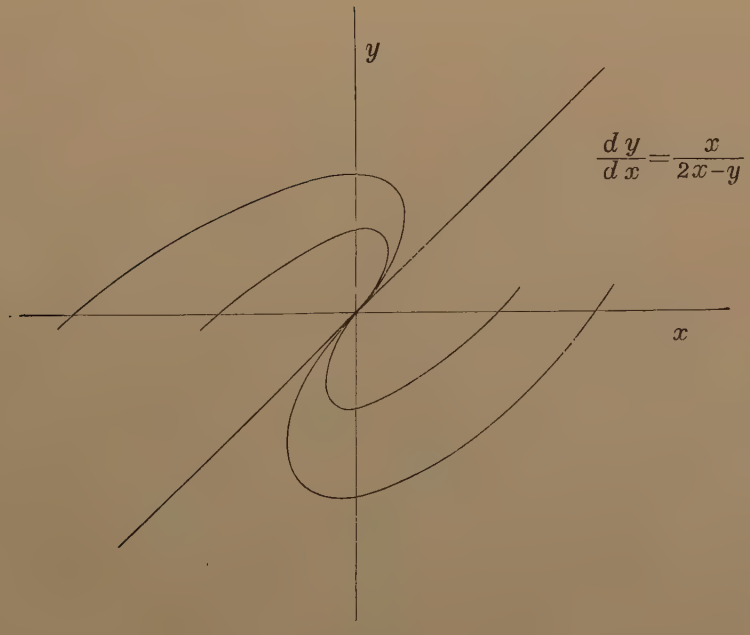


Fig. 29.

14. Examples.

The general principle for tracing characteristics defined by the equation $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ is due to the investigation of

1° $f(x,y) = \frac{P(x,y)}{Q(x,y)}$ i.e. a) curve $P(x,y)=0$, b) curve $Q(x,y)=0$ c) sign of $f(x,y)$;

2° $F(x,y)$ i,e. a) curve $F(x,y)=0$, b) sign of $F(x,y)$, c) $\chi\{(x,y)\}_{F=0} = \left(\frac{dy}{dx}\right)_{F=0} - (f(x,y))_{F=0}$;

3° Nature of singular points;

4° Nature of characteristics at infinite distance;

5° Limiting cycles.

Examples of tracing characteristics from their equations.

Ex. 1. $\frac{dy}{dx} = y^2 - x^2$ (Fig. 32).

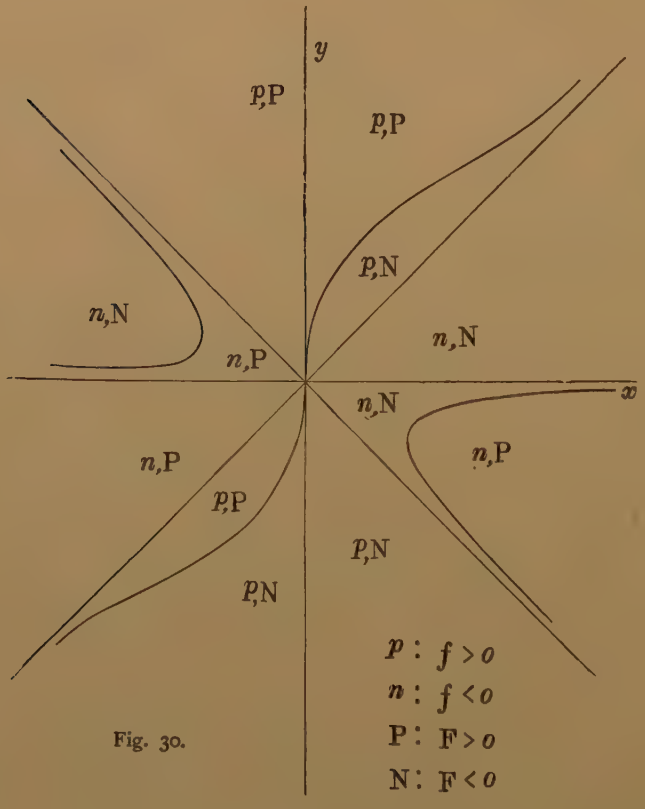


Fig. 30.

We have $f(x,y)=y^2-x^2$, $F(x,y)=2\{y(y^2-x^2)-x\}$

and the configuration of slope and concavity is shewn in Fig. 30.

On the curve $F(x,y)=0$,

$$[\chi(x,y)]_{F=0}=\left[\frac{y^2-x^2}{y^2(3y^2-x^2)}\right]_{F=0}$$

and we find

$$\chi>0 \quad \text{when} \quad 1<\frac{y}{x}<\infty \quad \text{and} \quad -\frac{1}{\sqrt{3}}<\frac{y}{x}<0,$$

$$\chi<0 \quad \text{when} \quad -1<\frac{y}{x}<-\frac{1}{\sqrt{3}},$$

$$\chi=\infty \quad \text{when} \quad \frac{y}{x}=-\frac{1}{\sqrt{3}}$$

and $[\chi(x,y)]_{F=0}=0$ has no root, so there is no contact point on the curve $F=0$.

The given equation has no singular point.

At infinite distance the possible directions are (1) $\alpha_1=\infty$, (2)

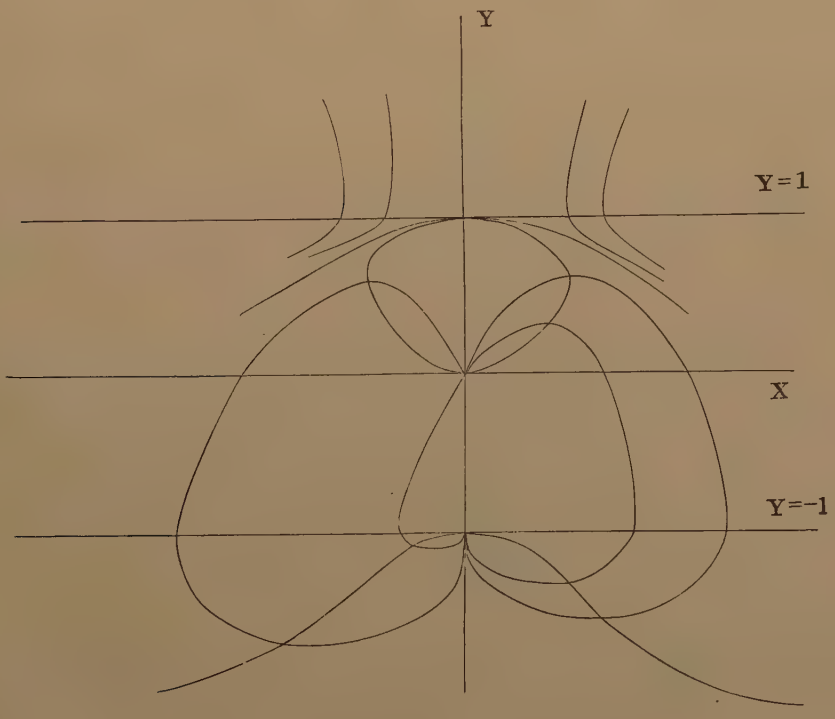


Fig. 31.

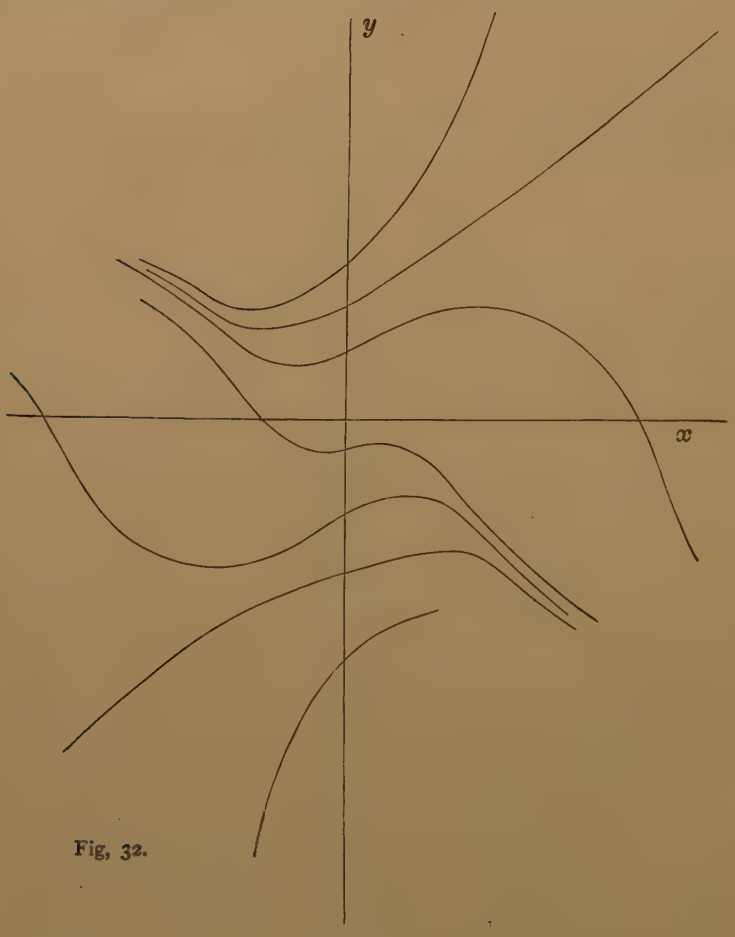
$a_2=1$ and (3) $a_3=-1$. By the substitution $x=\frac{Y}{X}$, $y=\frac{1}{X}$ the differential equation becomes

$$\frac{dY}{dX}=\frac{Y-Y^3-X^2}{X(1-Y^2)} \qquad \text{(Fig. 31)}$$

and the three directions go to the points: (1) $X=0$, $Y=0$; (2) $X=0$, $Y=1$ and (3) $X=0$, $Y=-1$.

(1) The point $X=0$, $Y=0$ is a node and in each direction there is a single characteristic.

(2) Put $Y=Y_1+1$, then $\frac{dY_1}{dX}=\frac{2Y_1+3Y_1^2+Y_1^3+X^2}{X(Y_1^2+2Y_1)}\equiv f_1(X,Y_1)$ and the possible directions at $X=0$, $Y_1=0$ are $\theta=0$, $\theta=\pi$, $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$. The integral $X=0$ is the only characteristic in the direction



Fig, 32.

$\theta=\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$. There is a single characteristic on each direction $\theta=0$, $\theta=\pi$.

(3) $X=0$, $Y=-1$ is a node and all the characteristics near the point have the direction $-\frac{\pi}{2}$ at $X=0$, $Y=-1$.

Thus the differential equation at infinite distance has (1) characteristics which have arbitrary straight lines parallel to the y-axis as asymptotes, (2) single characteristic on each of first and third quadrants, which has the line $x=y$ as asymptote, (3) indefinitely many characteristics which have the direction $-\frac{\pi}{4}$ in second and fourth quadrants.

Ex. 2. $$x\frac{dy}{dx}=x-2y+y^2 \quad \text{(Fig. 35).}$$

Here $$f(x,y)=\frac{x-2y+y^2}{x}$$

$$F(x,y)=\frac{-2x+6y+2xy-7y^2+2y^3}{x^2},$$

$$\{\chi(x,y)\}_{F=0}=\frac{y^2-2y}{(6-14y+13y^2-4y^3)(3-2y)}\Bigg\},$$ and the curve

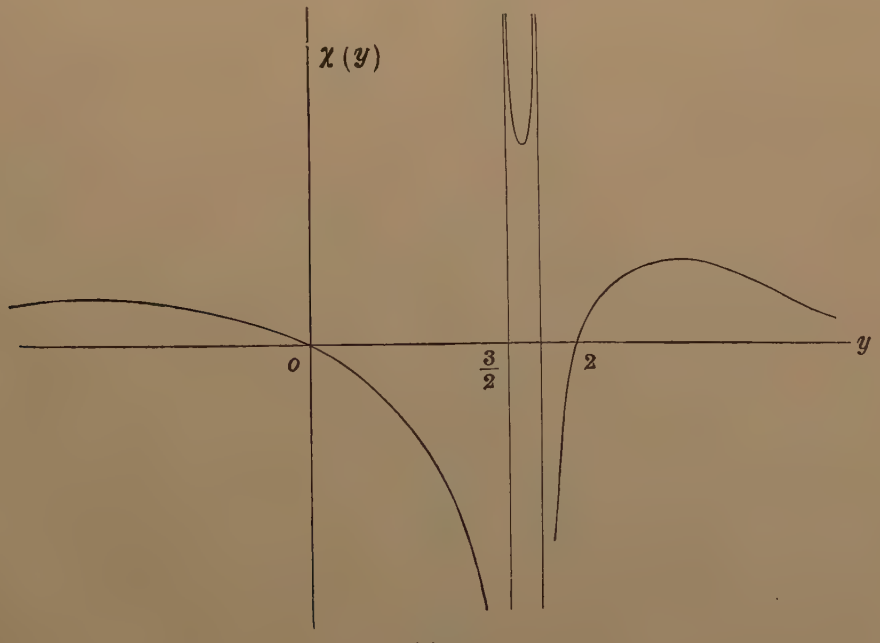


Fig. 33.

$\{\chi(x,y)\}_{F=0} = \chi(y)$ is shewn in Fig. 33.

The singular points are $x=0, y=0$ and $x=0, y=2$.

The origin is a *col* through which, besides the characteristic $x=0$, a characteristic with the slope $\frac{1}{3}$ passes.

The point $x=0, y=2$ is a node. At that point all characteristics except $x=0$ have the slope -1.

The possible directions at infinite distance are $a=\infty$ and $a=0$.

$a=\infty$. Put $x=\frac{Y}{X}, y=\frac{1}{X}$ then

$$\frac{dY}{dX} = \frac{Y(1-3X+XY)}{X(1-2X+XY)}.$$

Hence $X=0$, $Y=0$ is a node through which in each direction a single characteristic passes.

$a=0$. Put $x=\frac{1}{X_1}, y=\frac{Y_1}{X_1}$, then

$$\frac{dY_1}{dX_1} = \frac{-X_1+3X_1Y_1-Y_1^2}{X_1^2} \quad \text{(Fig. 34).}$$

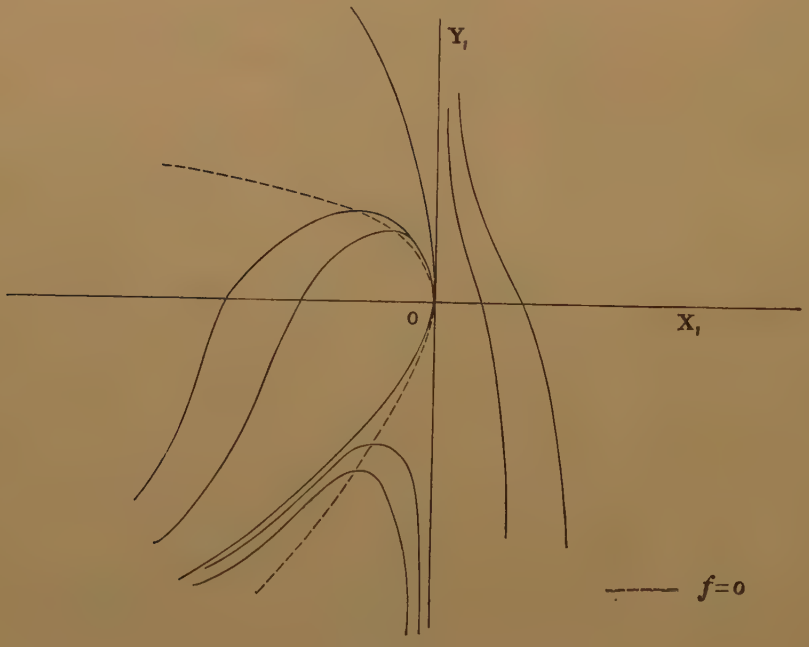


Fig. 34.

At $X_1=0$, $Y_1=0$ the possible direction is ∞. Using the configuration of slope and the criterions given in § 7, we find: $X_1=0$ is a characteristic; in the first and fourth quadrants there is no characteristic through the origin; in the third quadrant there is a single characteristic C_1 which has the slope $+\infty$ at the origin; the characteristics below C_1 are parabolic and those above C_1 are rays having the slope $-\infty$ at the origin.

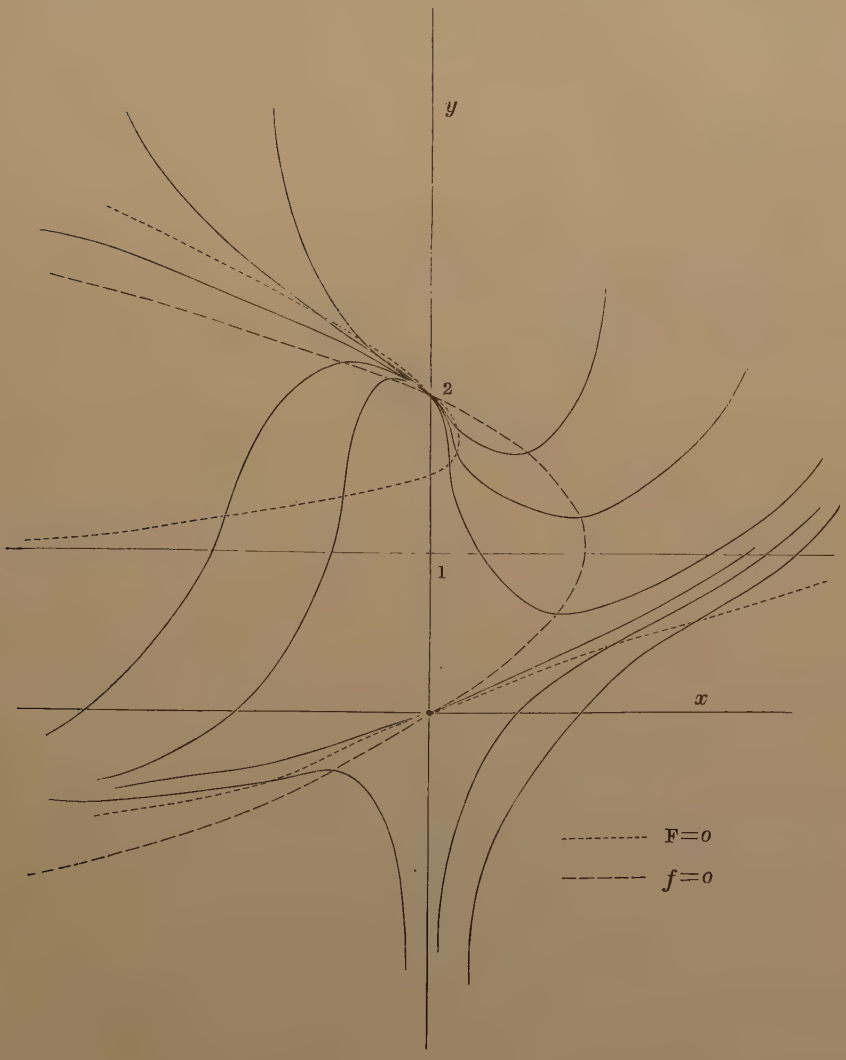


Fig. 35.

Hence of the original equation we may say: 1° there are two characteristics, one of which lies above the x-axis and the other below it, that have any line $x=c$ as asymptote; 2° in the second quadrant there is a single characteristic which has the slope 0 at infinity. This is the characteristic corresponding to C_1; 3° in the third quadrant there is an infinite number of characteristics which have the slope 0 at infinite distance.

Ex. 3. $$\frac{dy}{dx}=\frac{4x^3y}{y^2+x^4} \qquad \text{(Fig. 36).}$$

We have $f(x,y)=\frac{4x^3y}{y^2+x^4}$, $F(x,y)=\frac{4x^2y\,(3y^4-2x^4y^2+3x^8)}{(y^2+x^4)^3}$, $\frac{\partial f}{\partial y}=\frac{4x^3(x^4-y^2)}{(y^2+x^4)^2}$ and we know that $f(x,y)$, $F(x,y)$ are positive or negative according as $\frac{y}{x}$ is positive or negative.

The equation has only one singular point $x=0$, $y=0$ and evidently $y=0$ is a characteristic through it.

To find if there will be characteristics besides this, we apply criterion 4 § 7. Assume

$$\eta=a\xi^n,\quad n>1$$

then $$\chi(\xi,\eta)=\frac{d\eta}{d\xi}-f(\xi,\eta)=na\xi^{n-1}-\frac{4a\xi^{n+3}}{a^2\xi^{2n}+\xi^4}$$

and we have, if $\xi>0$,

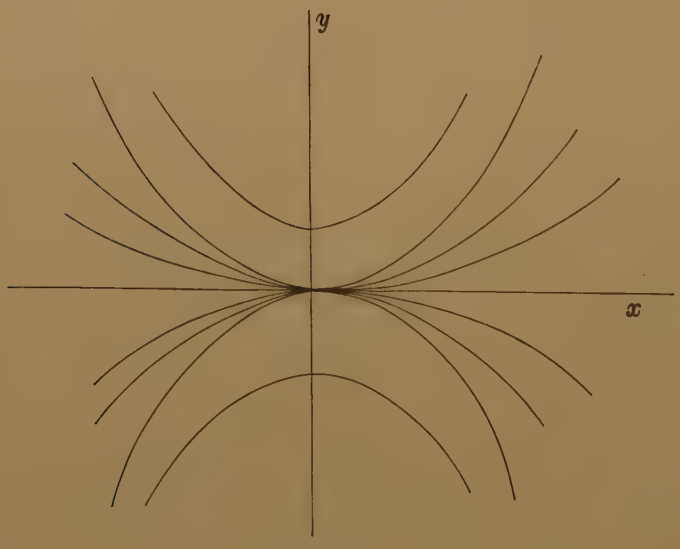


Fig. 36.

i) $\chi > 0$ when $1 < n < 2$, $a > 0$ or $n = 2$, $a > 1$;

ii) $\chi = 0$ when $n = 2$, $a = 1$;

iii) $\chi < 0$ when $n = 2$, $0 < a < 1$ or $2 < n < 4$, $a > 0$;

iv) $\chi > 0$ when $n > 4$, $a > 0$.

By ii) $y = x^2$ is a characteristic and, since $\frac{\partial f}{\partial y} < 0$, above this curve there is no characteristic through the origin. In ($0 \leqq y \leqq x^2$, $x > 0$) all characteristics pass through the origin and touch the positive x-axis. Also we know that the characteristics are symmetrical with respect to the x or the y-axis.

The possible directions at infinite distance are $a = 0$ and $a = \infty$; and all the characteristics except $y = 0$ have the slope $+\infty$ at infinity.

Ex. 4. $$\frac{dy}{dx} = \frac{-x - y + y(x^2 + y^2)}{y}$$ (Fig. 37).

The configuration of slope is shewn in the figure. The equation has only one singular point $x = 0$, $y = 0$; and this point is a focus.

Consider the circle

$$\xi^2 + \eta^2 = \rho^2$$

then $$\chi(\xi, \eta) = \frac{d\eta}{d\xi} - f(\xi, \eta) = 1 - \rho^2,$$

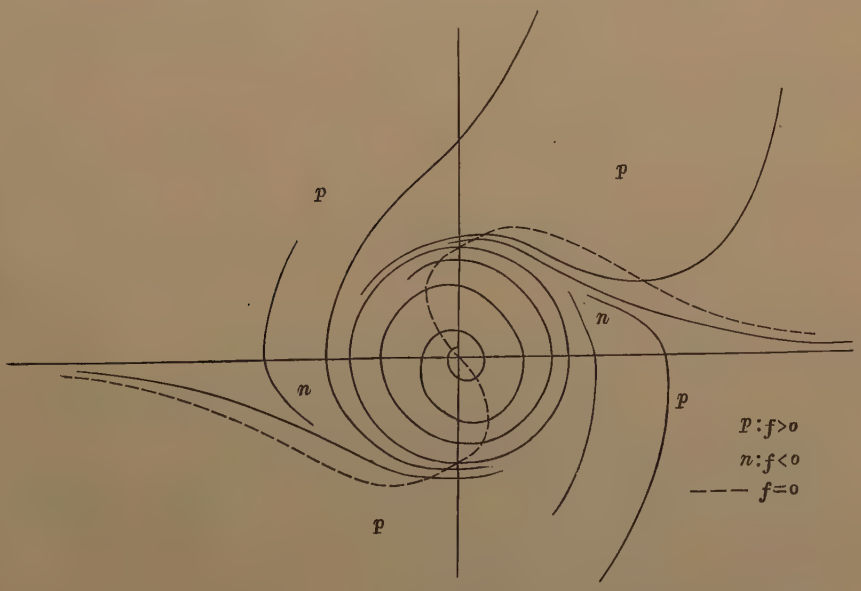


Fig. 37.

therefore
$$\chi(\xi, \eta) > 0 \quad \text{when} \quad 0 < \rho < 1,$$
$$= 0 \quad \text{when} \quad \rho = 1,$$
$$< 0 \quad \text{when} \quad 1 < \rho.$$

Thus the circles $\xi^2+\eta^2=\rho^2$ have no contact points except the points $\xi=\pm\rho, \eta=0$ when $\rho \neq 1$. At the contact points the circle and the characteristics cut each other and we see that the circle $\xi^2+\eta^2=1$ is the only limiting cycle.

In each of the first and third quadrants there is a characteristic which has the x-axis as asymptote. All the other characteristics have the slope ∞ at infinite distance.

Ex. 5
$$x\{2(x^2+y^2)-1\}dx+$$
$$[\{2(x^2+y^2)+1\}y+\frac{3}{2}(2x^2-1)\{(x^2+y^2)^2-x^2+y^2\}]dy=0 \quad \text{(Fig. 38).}$$

The equation has four finite singular points $x=0, y=0$; $x=0, y=1$; $x=\frac{1}{\sqrt{2}}, y=0$ and $x=-\frac{1}{\sqrt{2}}, y=0$.

1° $x=0, y=0$ is a *col* and the slopes of the characteristics through that point are -1 and $+1$.

2° $x=0, y=1$ is also a *col* and the slopes of the characteristics through that point are $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}$.

3° Put $x=\frac{1}{\sqrt{2}}+X$, then
$$\frac{dy}{dX}=\frac{-2X+\dots}{2y-\frac{3\sqrt{2}}{8}X+\dots}$$

and we know that the point $x=\frac{1}{\sqrt{2}}, y=0$ is a focus.

The equation remains unchanged by putting $-x$ for x, hence the line-elements are symmetrical with respect to the y-axis.

The equation has the characteristic $(x^2+y^2)^2-x^2+y^2=0$; and each loop of this lemniscate is the limiting cycle of the spirals which have the foci at $x=\frac{1}{\sqrt{2}}, y=0$ or $x=-\frac{1}{\sqrt{2}}, y=0$.

By the configuration of the slope the form of the characteristics may be traced as shewn in Fig. 38.

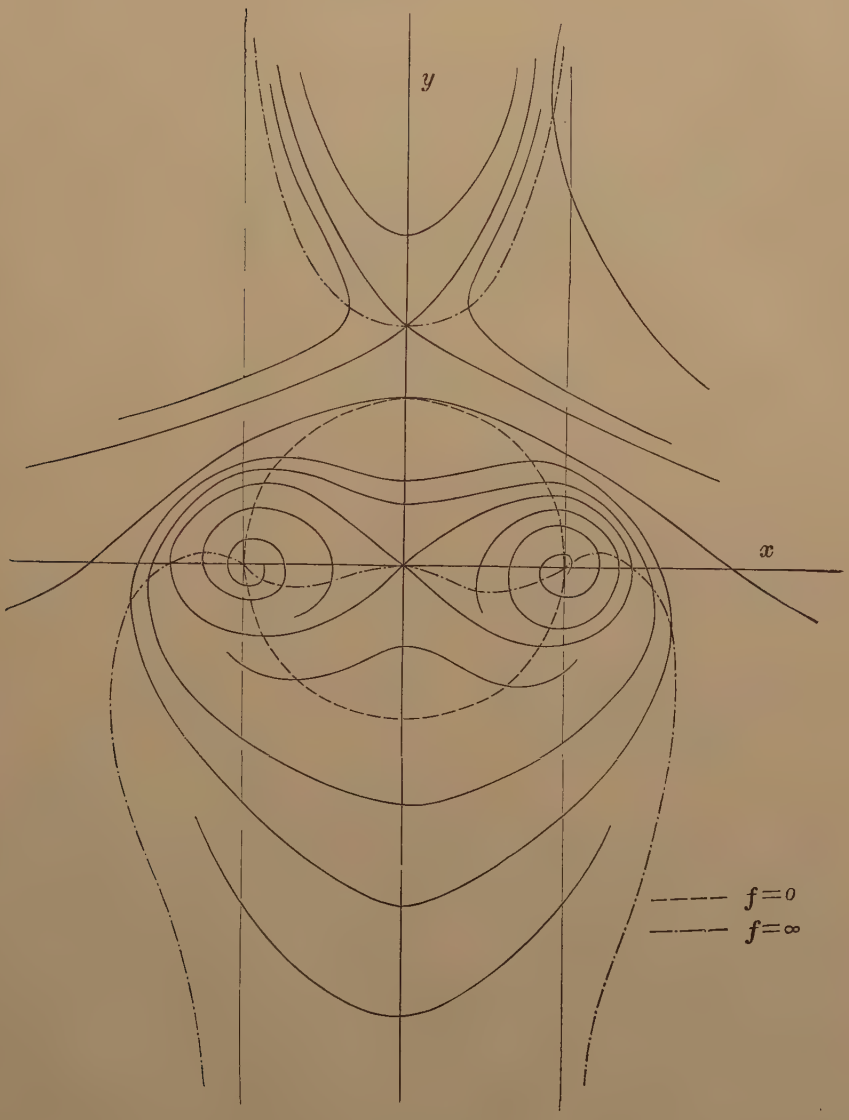


Fig. 38.

On the Absorption Spectrum of Oxyhæmoglobin in the Ultra-violet and Extreme Ultra-violet Region.

By

Toshikazu Mashimo.

(Received March 8, 1917.)

In order to investigate the molecular constitution of a substance, its physical and chemical properties are utilized. Among physical properties, the spectrum affords us a convenient and sensitive means for attacking the subject. For a liquid substance its absorption spectrum is used, if it has any selective absorption. Sometimes it is possible to ascertain the nature and the structure of the molecule of the substance by its absorption spectrum.

Hæmoglobin in the red blood corpuscles, which plays a great rôle in human metabolism, has been investigated physically and chemically by many; and its absorption spectrum has been exhaustively examined. By adding oxygen gas, hæmoglobin can be changed into oxyhæmoglobin; and the latter can be reversed to the former by means of ammonium sulphide. Oxyhæmoglobin easily decomposes to its derivatives. As each of those derivatives gives its characteristic absorption band, this change can be easily traced by the spectroscopic method. Oxyhæmoglobin and some of its derivatives in very dilute solutions have each an absorption band at $\lambda 410\mu\mu$; and this absorption band is recommended in forensic medicine and internal medicine for the proof of the existence of blood.

With regard to the ultra-violet absorption, many investigations have been made. J. L. Soret[1] first discovered three absorption bands of oxyhæmoglobin in this region, and determined their positions as

1 J. L. Soret: Arch. sci. phys. et natur. (Geneve), **61**, 322 (1878), **66**, 429 (1883).

follows: the first begins from the Fraunhofer line G and extends beyond the line H, the second at Cd-line λ3247 Å.U. and the third at Cd-line λ 2743 Å.U. A. Gamgee[1] examined for these bands, but the band at Cd-line λ 3247 Å.U. was not observed; while Dhéré[2] found two at λ 4156 Å.U. and λ 2801–2665 Å.U., but could not observe any near Cd-line λ 3247 Å.U.

R. Hiller[3] and H. Grabe[4] also investigated this subject. Recently E. Peyrega and F. Vlès[5] have noticed that the band which had been found by Soret at Cd-line λ 3247 Å.U. really exists, though it is not at Cd-line λ 3247 Å.U., but at λ 333 $\mu\mu$. Before reading the Peyrega and Vlès' paper the writer examined the absorption spectrum of oxyhæmoglobin and found a band at λ 350 $\mu\mu$, but not at λ 333 $\mu\mu$.

Method of Experiment.

(A) The ultra-violet region. The spectrograph used was a quartz-spectrograph of the type C, made by Hilger. By making use of a Baly's absorption tube with quartz windows, the thickness of the solution of oxyhæmoglobin through which the rays of light is to pass was changed. It was assumed that hydration of oxyhæmoglobin did not occur during the process of the experiment and that Beer's law relating to the thickness of absorbing layer and its concentration holds true. For the source of light the positive crater of a 90° carbon arc was used. In the investigation of absorption spectra, the iron arc is commonly recommended, for the reason that this arc emits many lines of nearly equal intensity and also is convenient for determining the wave lengths of absorption bands. But a merely slight inequality in the intensity of the lines renders the detection of a weak absorption band impossible; and for this reason a carbon arc, instead of an iron arc, was employed in the present investigation. The constancy of the electric current through the arc was checked by an amperemeter. The photographic plates employed were "Wratten M plate." The existence of a trace of methæmoglobin, found during the spectroscopic treatment,

1 Zs. Biol, **34**, 505 (1896).

2 Compt. Rend. Soc. Biol., **61**, 718 (1906), Jahresber. Fortschr. Tierchem., **40**, 128 (1910).

3 Diss. Rostock., (1904), Jahresber. Fortschr. Tierchem., **35**, 140 (1905).

4 H. Grabe: Diss. Dorpat., (1892).

5 C. R., **154**, 133 (1912).

might be detected by its characteristic strong absorption band in the visible part of the spectrum. But methæmoglobin does not give an absorption band in the region of the present investigation. The oxyhæmoglobin was prepared by the Hoppe Seyler's method from arterial blood which was taken by canule from a carotis of a dog; it was recrystallized three times, and then, without drying, the crystal was dissolved in distilled water. By this manipulation the absolute quantity of the oxyhæmoglobin could not be determined; but for the purpose this was not necessary. Many photographs were taken with the solution of the same degree of dilution which was estimated by comparison with the standard colour of Gower's hæmoglobinometer.

(B) The Schumann region. To obtain the spectrum of the extreme ultra-violet i.e. of the Schumann region, a quartz spectrograph of the type A of Hilger was used. As the source of light, a condensed spark of Co, Fe, or Wo in the air was used. In order to avoid the absorption due to the gelatin of the photographic plate; Schumann plates were taken. And also in order to get rid of the absorption of the water layer in this region of the spectrum as much as possible, 1 mm thick cuvette with quartz windows was used for the absorption vessel. The solutions of the different concentrations were examined.

Result.

In the annexed photograph, the white lines represent the emission lines. The thickness of the liquid layer was changed in the ratio of the numbers 1 : 2 : 3 etc. up to 42. The 43rd spectrum is that of the source only. The band seen at λ 350 $\mu\mu$ is probably a controversed absorption band, and is of course the one which Peyrega and Vlès found at λ 333 $\mu\mu$. By tracing the absorption curve for this band in the photograph and from the curvature of this curve, the centre of this band was estimated to be at λ 350 $\mu\mu$.

Any noticeable absorption band was not detected in the Schumann region.

Conclusion.

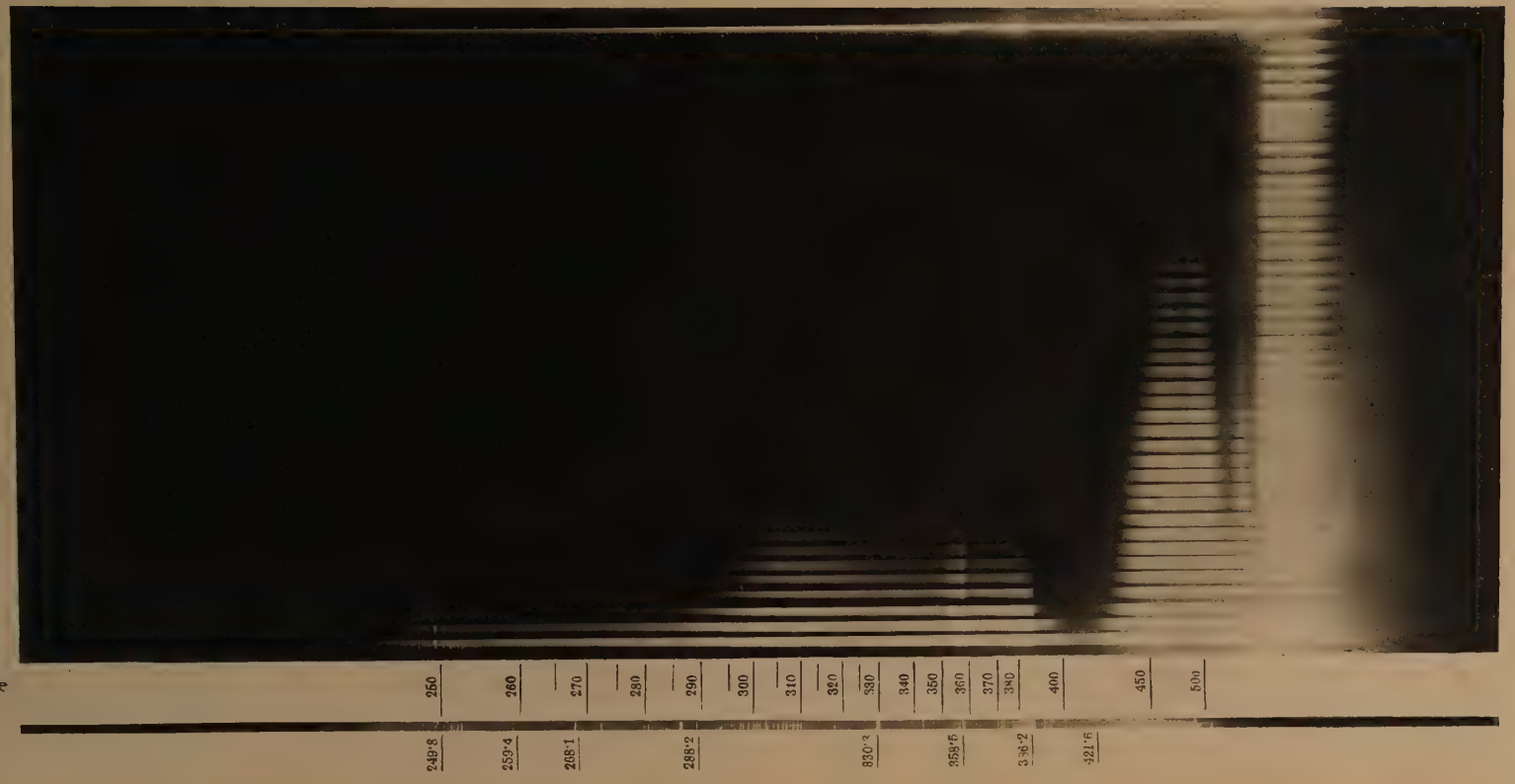
The disagreement among the observations on the absorption band of oxyhæmoglobin by many authors seems to be due to the fact that the concentration employed were not appropriate, as was pointed out by Peyrega and Vlès in the paper before cited. The writer took 42 degrees of concentration, and determined the position of the centre of the absorption band from the absorption curve thus obtained, and found

an absorption band at λ 350 $\mu\mu$, which may perhaps be the one observed by Peyrega and Vlès at λ 333 $\mu\mu$.

In the Schumann region no absorption band was observed.

The writer's sincere thanks are due to Prof. Mizuno, and Assist. Prof. Takamine for their kind suggestions, and also to Assist. Prof. Yamaguchi for his aid in preparing the oxyhæmoglobin.

Absorption Spectrum of Oxyhæmoglobin
in Ultra-violet Region.



On Congruences.

By

Masazo Sono.

(Received March 13, 1917).

Objects to be treated in mathematics are sets of elements which contain definitions of equality and of composition, as well as postulates concerning the rules of composition. Is it possible in a given object, completely defined, to change the definition of equality, under the conditions that elements equal to one another remain equal after the change and that the object only reduces to another of the same kind? If so, in what ways can it be done? In the present paper we give the answer to this in the case of groups and of rings; we investigate the possibility and the conditions for changing the definition of equality, under the restriction that a given group or a given ring remains also a group or a ring after the change. We add also a few theorems on rings.

It is desirable to have this problem solved also for the treatment of the cases where the elements of a group or a ring are variables, independent or dependent.

Definitions of a Group and a Ring.

§ 1. To build up an object of mathematical research, we first define the equality of two elements, next the composition of the elements, and lastly give the postulates concerning the rules of composition.

I. Equality.[1] The equality and the inequality of two elements belonging to a set are determined by a definition which consists of the two statements; the one explains that the two elements are equal; the other that they are unequal. For its establishment there is no further restriction than that the given definition must satisfy the following conditions:

---

1 O. Stolz and J. A. Gmeiner, Theoretische Arithmetik, p. 2.

1. Every element is equal to itself in consequence of the definition. If A is equal to B, B must be equal to A.

2. Two elements of the set are either equal or unequal.

3. From $A=B$ and $B=C$ follows $A=C$.

II. Composition. It is said that a *composition* upon the elements A and B of a set is defined, if, corresponding to the elements A and B and to a certain order of these elements, there exists a certain third elements C, which may or may not belong to the set. The new element C which is associated with the given elements in the given order, is called the *result* of the composition; the composition of A and B is written AB when the order of the elements is A, B. For convenience the symbol AB is considered to denote the result of the composition at the same time. Consequently, if the result of the composition of A and B is C, we have $AB=C$.

A composition is said to be *unique*[1] when from $A'=A$ and $B'=B$ follows $A'B'=AB$. In our case, groups and rings, the composition is defined as always unique. If there are two rules in regard to the composition of two elements A and B, say denoted by $+$ and $\times$, we distinguish them one from the other in writing by $A+B$ and $A\times B$.

III. The postulates for

(a) Group:[2] A set $\mathfrak{G}$ of elements, in which the composition among its elements is defined, is called a *group* with respect to the given composition when the following postulates hold:

1. If A and B are elements of $\mathfrak{G}$, AB is also an element of $\mathfrak{G}$.

2. If A, B and C are three elements of $\mathfrak{G}$, the result of the composition of A and BC is equal to that of AB and C. That is, $A(BC)=(AB)C$. This is the so-called associative law.

3. There exists in $\mathfrak{G}$ an element E, such that $AE=A$, for every element A of $\mathfrak{G}$.

4. There exists in $\mathfrak{G}$, corresponding to any element A of $\mathfrak{G}$, another A^{-1} such that $AA^{-1}=E$, where E is the said element.

(b) Ring: A set $\mathfrak{R}$ of elements with two rules of composition denoted by $+$ and $\times$ is called a *ring* when the following nine postulates hold:

1. If A and B are two elements of $\mathfrak{R}$, $A+B$ is also an element of $\mathfrak{R}$.

2. If A and B are elements of $\mathfrak{R}$, then $A+B=B+A$. (Commutative law).

---

[1] *Loc. cit.*, p. 4. G. Frobenius, Berliner Sitzber. (1895), p. 163.

[2] L. E. Dickson, Trans. Amer. Math. Soc., **6**, 199 (1905).

3. If A, B and C are three elements of $\mathfrak{R}$, then $(A+B)+C=A+(B+C)$. (Associative law).

4. There exists in $\mathfrak{R}$ an element Z, such that $Z+B=B$, for every element B of $\mathfrak{R}$.

5. There exists in $\mathfrak{R}$, corresponding to every element A of $\mathfrak{R}$, another X such that $A+X=Z$, where Z is the said element.

6. If A and B are two elements of $\mathfrak{R}$, $A\times B$ is also an element of $\mathfrak{R}$.

7. If A and B are two elements of $\mathfrak{R}$, then $A\times B=B\times A$. (Commutative law).

8. If A, B and C are elements of $\mathfrak{R}$, then $(A\times B)\times C=A\times(B\times C)$. (Associative law).

9. $A\times(B+C)=(A\times B)+(A\times C)$. (Distributive law).

We have defined a ring abstractly by the above nine postulates after consideration of the points which are essential in rings.[1] This set of postulates is a part of the set by which Prof. Dickson[2] has defined a field abstractly; hence there is no necessity of again showing the independency of the postulates.

N. B. The set $\mathfrak{R}$ becomes a field[3], when the following two postulates hold in addition to the above nine:

(i) There exists in $\mathfrak{R}$ an element U such that $U\times B=B$ for every element B of $\mathfrak{R}$.

(ii) Corresponding to every element A such that $C\times A\neq A$ for at least one element C of $\mathfrak{R}$, there exists in $\mathfrak{R}$ an element X for which $A\times X=U$, where U is the said element.

We now proceed to change the definition of equality under the restriction that elements equal to one another remain equal after the change and that a group or a ring remains also a group or a ring after the change, that is, that the conditions for equality are satisfied, the composition remains unique and moreover the postulates for a group or a ring hold.

When an element A becomes equal to another B by changing the definition of equality, we say for a time to avoid ambiguity that A is *congruent* to B, and this is expressed in writing by $A\equiv B$.

---

1 Hilbert, Jahresber. D. Math. Ver. **4**, 237, (1894–95).
Dedekind used the word *Ordnung* for *Ring*; and he defined it as a modulus $\mathfrak{M}$ such that it contains the square $\mathfrak{M}^2$ and also the number 1. But, for some reason, we have omitted the second condition.

2 Dickson, Trans. Amer. Math. Soc., **4**, 17, (1903).

3 *Loc. cit.*

Since in the change considered the conditions for equality and the uniqueness of the composition hold, we have that from $A' \equiv A$, $B' \equiv B$ and $AB = C$ follows $A'B' \equiv C$.

Congruence in a Group.

§ 2. Let $\mathfrak{G}$ be a given group; and suppose that the definition of equality has been so changed that $\mathfrak{G}$ remains a group. Let $\mathfrak{H}$ be the set formed of all the elements of $\mathfrak{G}$ which become congruent to the identity E of $\mathfrak{G}$ after the change of the definition of equality. For any two elements H_1 and H_2 of $\mathfrak{H}$ we have

$$H_1 H_2 \equiv EE,$$

and

$$EE = E,$$

since $H_1 \equiv E$, $H_2 \equiv E$ and the composition is unique; therefore, the product of two elements of $\mathfrak{H}$ is also congruent to the identity E of $\mathfrak{G}$ and consequently belongs to $\mathfrak{H}$. Hence the set $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$. Let A and H be elements of $\mathfrak{G}$ and $\mathfrak{H}$ respectively, then we have

$$A^{-1}HA \equiv A^{-1}EA \equiv E,$$

since elements equal to one another become congruent after the change and the composition is unique; therefore the element $A^{-1}HA$ belongs to $\mathfrak{H}$, that is, $\mathfrak{H}$ is self-conjugate under $\mathfrak{G}$. Thus the elements which become congruent to the identity of $\mathfrak{G}$ by the change of the definition of equality constitute a self-conjugate subgroup $\mathfrak{H}$ of $\mathfrak{G}$.

If an element B of $\mathfrak{G}$ becomes congruent to another A by the said change, i.e. $B \equiv A$, then, since $A^{-1} \equiv A^{-1}$ and the composition is unique, we have

$$BA^{-1} \equiv AA^{-1} \equiv E,$$

which shows that BA^{-1} belongs to $\mathfrak{H}$ and therefore B is an element of the co-set $\mathfrak{H}A$ of $\mathfrak{G}$ as regards $\mathfrak{H}$. If, conversely, B be an element of the co-set $\mathfrak{H}A$, B is of the form HA, where H denotes an element of $\mathfrak{H}$, and we have

$$B \equiv HA \equiv EA \equiv A,$$

since $H \equiv E$. Therefore the elements which become congruent to A belong to the same co-set $\mathfrak{H}A$; and, conversely, each element of $\mathfrak{H}A$ becomes congruent to A. Hence we have the

THEOREM: *In order that such a change as we desire can be made the following conditions must be satisfied:*

(i) *The elements of a given group* $\mathfrak{G}$, *which become congruent to the identity of* $\mathfrak{G}$, *constitute a self-conjugate subgroup* $\mathfrak{H}$ *of* $\mathfrak{G}$.

(ii) *The elements of the same co-set of* $\mathfrak{G}$ *as regards* $\mathfrak{H}$ *become congruent to one another.*

(iii) *The elements congruent to one another belong to the same co-set of* $\mathfrak{G}$ *as regards* $\mathfrak{H}$.

And, conversely, these conditions are sufficient.

Let $\mathfrak{H}$ be a self-conjugate subgroup of $\mathfrak{G}$ and suppose that the change has been made to satisfy the above three conditions. Then the conditions for equality are evidently satisfied. If H_1 and H_2 be two elements of $\mathfrak{H}$ and if A and B be two elements of $\mathfrak{G}$, we have

$$H_1A \cdot H_2B = HAB,$$

where H is an element of $\mathfrak{H}$; and therefore

$$H_1A \cdot H_2B \equiv A \cdot B,$$

which shows that the uniqueness of the composition holds. And, moreover, evidently the postulates for a group also hold.

§ 3. We see from the result obtained above that we can change completely the definition of equality, as we desire, by putting the elements of a self-conjugate subgroup $\mathfrak{H}$ of a given group $\mathfrak{G}$ congruent to the identity of $\mathfrak{G}$, and only by so doing. We, therefore, define again:

When, the definition of equality in a given group $\mathfrak{G}$ being so changed that each element of a self-conjugate subgroup $\mathfrak{H}$ becomes equal to the identity of $\mathfrak{G}$, two elements A and B of $\mathfrak{G}$ become equal to each other, A and B are said to be *congruent*[1] with respect to the modulus $\mathfrak{H}$, and this is expressed in writing by

$$A \equiv B \pmod{\mathfrak{H}}.$$

The group, to which the given group $\mathfrak{G}$ reduces when we take its elements with respect to the self-conjugate subgroup $\mathfrak{H}$, is easily shown to be simply isomorphic with the quotient group $\frac{\mathfrak{G}}{\mathfrak{H}}$; this is available to define the quotient: the quotient $\frac{\mathfrak{G}}{\mathfrak{H}}$ may be considered for the group, to which $\mathfrak{G}$ reduces when its elements are taken with respect to the modulus $\mathfrak{H}$. This view will be found useful in simply solving certain problems concerning quotient groups.

---

1 The term *congruent* is sometimes used in the same meaning.

Simple Applications.[1]

§ 4. (1) If each element of a group $\mathfrak{H}$ is permutable with another group $\mathfrak{K}$, the groups $\{\mathfrak{H}, \mathfrak{K}\}/\mathfrak{K}$ and $\mathfrak{H}/\mathfrak{L}$ are simply isomorphic, where $\mathfrak{L}$ is the subgroup common to $\mathfrak{H}$ and $\mathfrak{K}$.

$\mathfrak{K}$ and $\mathfrak{L}$ are evidently self-conjugate subgroups of $\{\mathfrak{H}, \mathfrak{K}\}$ and $\mathfrak{H}$ respectively. The elements of $\{\mathfrak{H}, \mathfrak{K}\}$ being taken with respect to the modulus $\mathfrak{K}$, each element of $\mathfrak{K}$ becomes $\equiv 1 \pmod{\mathfrak{K}}$; and the elements in $\mathfrak{H}$ which become congruent to $1 \pmod{\mathfrak{K}}$ are those of $\mathfrak{L}$. If two elements A and B of $\mathfrak{H}$ are incongruent $(\mathrm{mod.}\ \mathfrak{L})$, A and B are also incongruent $(\mathrm{mod.}\ \mathfrak{K})$. Indeed, if $A \equiv B \pmod{\mathfrak{K}}$, then $AB^{-1} \equiv 1 \pmod{\mathfrak{K}}$, that is AB^{-1} would belong to $\mathfrak{K}$, while it belongs to $\mathfrak{H}$. Therefore AB^{-1} must belong to $\mathfrak{L}$, i.e. $AB^{-1} \equiv 1 \pmod{\mathfrak{L}}$, whence $A \equiv B \pmod{\mathfrak{L}}$. But this contradicts the assumption that $A \not\equiv B \pmod{\mathfrak{L}}$. Therefore now let

$$A,\ B,\ C,\ \ldots\ldots \pmod{\mathfrak{L}}$$

be a complete set of incongruent $(\mathrm{mod.}\ \mathfrak{L})$ elements of $\mathfrak{H}$, which is $\mathfrak{H}/\mathfrak{L}$, then the elements

$$A,\ B,\ C,\ \ldots\ldots \pmod{\mathfrak{K}},$$

being taken with respect to the modulus $\mathfrak{K}$, form a complete set of incongruent $(\mathrm{mod.}\ \mathfrak{K})$ elements of $\{\mathfrak{H}, \mathfrak{K}\}$, which is the group $\{\mathfrak{H}, \mathfrak{K}\}/\mathfrak{K}$. If a correspondence between the elements of $\{\mathfrak{H}, \mathfrak{K}\}/\mathfrak{K}$ and those of $\mathfrak{H}/\mathfrak{L}$ be established, so that to $A, B, C, \ldots \pmod{\mathfrak{K}}$ of the former there correspond respectively $A, B, C, \ldots \pmod{\mathfrak{L}}$ of the latter, then to the product $AB \pmod{\mathfrak{K}}$ of any two elements A and B of $\{\mathfrak{H}, \mathfrak{K}\}/\mathfrak{K}$ there corresponds the product $AB \pmod{\mathfrak{L}}$ of the corresponding elements of $\mathfrak{H}/\mathfrak{L}$. Hence the two groups are simply isomorphic. For, if $AB \equiv C \pmod{\mathfrak{K}}$, then $ABC^{-1} \equiv 1 \pmod{\mathfrak{K}}$ and hence $ABC^{-1} \equiv 1 \pmod{\mathfrak{L}}$, since ABC^{-1} belongs to $\mathfrak{H}$. Consequently $AB \equiv C \pmod{\mathfrak{L}}$. If, conversely, $AB \equiv C \pmod{\mathfrak{L}}$, we evidently have $AB \equiv C \pmod{\mathfrak{K}}$.

(2) If A and B are two elements of a group $\mathfrak{G}$, $A^{-1}B^{-1}AB$ is the commutator of A and B; it is a well-known fact that the commutator-group of a group is self-conjugate. If we now put $A^{-1}B^{-1}AB \equiv 1$, we have $AB \equiv BA$, which shows that the quotient of $\mathfrak{G}$ by the commutator-group is abelian. Conversely, if the quotient of $\mathfrak{G}$ by a self-conjugate subgroup $\mathfrak{H}$ is abelian, i.e.

[1] The facts stated in (1) and (2) are well known, but the author here gives simple proofs of them.

$$AB \equiv BA \pmod{\mathfrak{H}},$$

it follows that

$$A^{-1}B^{-1}AB \equiv 1 \pmod{\mathfrak{H}};$$

hence $\mathfrak{H}$ must contain the commutator-group.

(3) Words of application to permutation-groups. It is not necessary to consider that the letters on which a permutation are performed are all distinct: the operation of replacing each letter of a given set of letters by another, which may or may not be equal, when carried out under the condition that no two unequal letters are replaced by letters equal to each other, is called a permutation. When the equality between the letters on which the permutations of a permutation-group are performed are so changed that each permutation of the group remains also a permutation, the group reduces to another having fewer different letters. This fact somewhat facilitates certain discussions of intransitive and imprimitive groups.

Congruence in a Ring.

§ 5. Before considering congruences in a ring, we should mention a few properties of a ring which follow immediately from the postulates.

The first five of the postulates for a ring show that every ring is an abelian group with respect to addition[1]. Hence it follows that the element Z stated in the fourth postulate is the same for all the elements of a ring $\mathfrak{R}$ and that there exists only one, and also that, corresponding to any given element A, there exists only one element X such that $A+X=Z$. In other words, if $Z+B=B$ for a certain element B of a ring $\mathfrak{R}$, then $Z+C=C$ for every element C of $\mathfrak{R}$; and if $Z'+B=B$, then $Z'=Z$; if $A+X'=Z$ and $A+X=Z$ for a given element A, then $X'=X$. Hereafter the said element Z is denoted by the symbol 0 and the said element X corresponding to A by the symbol $-A$. According to these notations we have $A+(-A)=0$. And also for two elements A and B of $\mathfrak{R}$ there is uniquely determined an element Y such that $A+Y=B$.

Since

$$A\cdot 0+A\cdot B=A(0+B)=A\cdot B=0+AB,$$

we have

---

1 The two rules of composition denoted by $+$ and $\times$ are called respectively addition and multiplication. For simplicity we write AB or $A\cdot B$ for $A\times B$.

$$A \cdot 0 = 0$$

for each element A of $\mathfrak{R}$.

Since for two elements A and B of $\mathfrak{R}$

$$A \cdot B + A(-B) = A(B+(-B)) = A \cdot 0 = 0,$$

and $$(-A)B+(-A)(-B) = (-A)(B+(-B)) = (-A) \cdot 0 = 0,$$

we have $$A(-B) = -(AB),$$

and $$(-A)(-B) = AB.$$

§ 6. It may happen that certain elements of a ring $\mathfrak{R}$ taken by themselves form a ring $\mathfrak{S}$. This ring $\mathfrak{S}$ is called a *subring* of $\mathfrak{R}$, and is said to be contained in $\mathfrak{R}$.

If a set $\mathfrak{S}$ of certain elements of $\mathfrak{R}$ satisfies postulates 1, 4, 5 and 6 of § 1, III, (b), it is evidently a subring of $\mathfrak{R}$. The simplest possible subring is that which consists of the element 0 alone.

Now let $\mathfrak{S}$ be a subring of a given ring $\mathfrak{R}$, and A be an element of $\mathfrak{R}$. Denote by the symbol $\mathfrak{S}+A$ the set consisting of all the elements of $\mathfrak{R}$, which are expressed in the form $S+A$ where S is an element of $\mathfrak{S}$. This set $\mathfrak{S}+A$ is called a *co-set* of $\mathfrak{R}$ as regards $\mathfrak{S}$.

If A belongs to the subring $\mathfrak{S}$, the co-set $\mathfrak{S}+A$ evidently coincides with $\mathfrak{S}$; but if not, $\mathfrak{S}$ and $\mathfrak{S}+A$ have no common element. Two co-sets $\mathfrak{S}+A$ and $\mathfrak{S}+B$ either are identical or have no common element. Indeed, if $S_1+A=S_2+B$, where S_1 and S_2 are elements of $\mathfrak{S}$, we have

$$A = (S_2+(-S_1))+B$$

and, consequently, A belong to $\mathfrak{S}+B$; and similarly, B also belongs to $\mathfrak{S}+A$. Therefore $\mathfrak{S}+A$ and $\mathfrak{S}+B$ are identical. Therefore *if* $\mathfrak{S}$ *is a subring of the ring* $\mathfrak{R}$, *the elements of* $\mathfrak{R}$ *are divided into co-sets of* $\mathfrak{R}$ *as regards* $\mathfrak{S}$, *such that every element of* $\mathfrak{R}$ *belongs to one, and only one, of the co-sets.*

§ 7. The changeability of the definition of equality in a ring, under the conditions stated in § 1. Let $\mathfrak{R}$ be a ring; and suppose that the definition of equality has been changed so that $\mathfrak{R}$ remains also a ring. Let $\mathfrak{M}$ be the set formed by all the elements of $\mathfrak{R}$ which become congruent to the element 0 of $\mathfrak{R}$ after the change considered. If an element M belongs to $\mathfrak{M}$, the element $-M$ also belongs to $\mathfrak{M}$. For, since

$$M \equiv 0 \qquad \text{(by the assumption)}$$

and $$-M \equiv -M,$$

we have

$$M+(-M) \equiv 0+(-M) \qquad \text{(by the unique composition)},$$

or

$$0 \equiv -M;$$

therefore $-M$ belongs to $\mathfrak{M}$. Next, let M_1 and M_2 be any two elements of $\mathfrak{M}$, and A an element of $\mathfrak{R}$; then we have

$$M_1+M_2 \equiv 0+0 \equiv 0,$$

$$M_1+(-M_2) \equiv 0+0 \equiv 0,$$

and

$$A\cdot M_1 \equiv A\cdot 0 \equiv 0.$$

Therefore, $\mathfrak{M}$ is a subring of $\mathfrak{R}$ such that every element of $\mathfrak{M}$ multiplied by an element of $\mathfrak{R}$ also belongs to $\mathfrak{M}$. This set $\mathfrak{M}$ is nothing but Dedekind's *ideal*[1], which is considered in the ring $\mathfrak{R}$, abstractly defined.

We here define an ideal in an abstract ring as follows:

A set $\mathfrak{M}$ of elements of a ring is called an *ideal* of the ring when it satisfies the following three conditions:

1. The sum of any two elements of $\mathfrak{M}$ is also an element of $\mathfrak{M}$.
2. If an element M belongs to $\mathfrak{M}$, $-M$ also belongs.
3. The product of any element of $\mathfrak{M}$ and any element of $\mathfrak{R}$ is an element of $\mathfrak{M}$.

In other words, an ideal $\mathfrak{M}$ is a subring of $\mathfrak{R}$ such that every element multiplied by an element of $\mathfrak{R}$ also belongs to $\mathfrak{M}$.

Returning to the subject in question, if A be an element of $\mathfrak{R}$, the elements of the co-set $\mathfrak{M}+A$ of $\mathfrak{R}$ as regards $\mathfrak{M}$ become congruent to one another by the change considered, because of the unique composition. When an element B of $\mathfrak{R}$ becomes congruent to another A by the change, i.e. $B \equiv A$, we have

$$B+(-A) \equiv A+(-A) \equiv 0;$$

hence

$$B+(-A) = M,$$

or

$$B = M+A,$$

where M denotes an element of $\mathfrak{M}$; this shows that B belongs to the co-set $\mathfrak{M}+A$. Therefore the elements which become congruent to A all belong to the same co-set $\mathfrak{M}+A$; and, conversely, each element of the co-set $\mathfrak{M}+A$ becomes congruent to A. Hence we have the

1 Dirichlet-Dedekind, Vorlesungen über Zahlentheorie, 4. ed., § 177.

THEOREM: *A change of the definition of equality in a ring under the restriction stated in § 1 is possible when, and only when, the following three conditions are satisfied:*

(i) *The elements of a given ring* $\mathfrak{R}$, *which become congruent to the element* 0 *of* $\mathfrak{R}$, *form an ideal* $\mathfrak{M}$ *of* $\mathfrak{R}$.

(ii) *The elements of the same co-set of* $\mathfrak{R}$ *as regards* $\mathfrak{M}$ *become congruent to one another.*

(iii) *The elements congruent to one another belong to the same co-set of* $\mathfrak{R}$ *as regards* $\mathfrak{M}$.

To prove the sufficiency of these conditions, let $\mathfrak{M}$ be an ideal of $\mathfrak{R}$; and suppose that the change of the definition of equality is made as above. Then, the conditions for the equality are evidently satisfied. If A and B are two elements of $\mathfrak{R}$, and if M_1 and M_2 are two elements of $\mathfrak{M}$, then we have

$$(M_1+A)+(M_2+B) = (M_1+M_2)+(A+B),$$

and

$$(M_1+A)(M_2+B) = (M_1M_2+AM_2+BM_1)+AB,$$

which belong to the co-sets $\mathfrak{M}+(A+B)$ and $\mathfrak{M}+(AB)$ respectively; hence

$$(M_1+A)+(M_2+B) \equiv A+B,$$

and

$$(M_1+A)(M_2+B) \equiv AB.$$

These show the uniqueness of the composition; and it is easily shown that the postulates for a ring in §1, III hold.

§ 8. For such a change of the definition of equality as considered, an ideal must be introduced; and a given ideal determines the change completely, as desired. Therefore, we here define again.[1]

When, the definition of equality in a given ring $\mathfrak{R}$ being so changed that each element of an ideal $\mathfrak{M}$ of $\mathfrak{R}$ becomes equal to the element 0, two elements A and B become equal to each other, the elements A and B are said to be *congruent* with respect to the modulus $\mathfrak{M}$; and this is expressed in writing by

$$A \equiv B \pmod{\mathfrak{M}}.$$

The concept of congruence in arithmetic is nothing but a change of the definition of equality; and, hence, ideals must be naturally introduced as moduli in congruences.

---

[1] This definition coincides with what is usually adopted.

When the definition of equality is changed in a field, the set of the elements which become congruent to the element 0 must have the same properties as the ideal above stated, and must consequently contain all the elements of the field, if it contains any one element other than 0. Hence in a field no congruence is considered.

N.B. This definition of congruence is applicable also when the modulus $\mathfrak{M}$ is the ideal consisting of the element 0 alone; this is the case when the definition of equality is not changed.

Quotient Rings.

§ 9. Definitions. Let $\mathfrak{M}$ be an ideal of a ring $\mathfrak{R}$. The ring, to which $\mathfrak{R}$ reduces when we take the elements of $\mathfrak{R}$ with respect to the modulus $\mathfrak{M}$, is called the *quotient ring*[1] of $\mathfrak{R}$ as regards $\mathfrak{M}$; and it is represented by the symbol $\frac{\mathfrak{R}}{\mathfrak{M}}$.

The number of incongruent (mod. $\mathfrak{M}$) elements in the ring $\mathfrak{R}$ is called the *norm* of $\mathfrak{M}$ under $\mathfrak{R}$, as in the theory of ideals in algebraic number-fields.

A ring is said to be *composite* or *simple*, according as it does or does not possess at least one ideal other than the ring itself and that consisting of the element 0 alone.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals of a ring $\mathfrak{R}$; and suppose that $\mathfrak{B}$ contains $\mathfrak{A}$. Then the ideal $\mathfrak{A}$ of $\mathfrak{R}$ is evidently also an ideal of $\mathfrak{B}$; and a complete set of incongruent (mod. $\mathfrak{A}$) elements of $\mathfrak{B}$, that is, the quotient ring $\frac{\mathfrak{B}}{\mathfrak{A}}$, is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$. For, if B_1 and B_2 (mod. $\mathfrak{A}$) be two elements of this set, and R (mod. $\mathfrak{A}$) an element of $\frac{\mathfrak{R}}{\mathfrak{A}}$, then we have

$$B_1+B_2 \equiv 0 \pmod{\mathfrak{B}},$$
$$B_1+(-B_2) \equiv 0 \pmod{\mathfrak{B}},$$

and

$$RB_1 \equiv 0 \pmod{\mathfrak{B}};$$

and moreover each of these three elements B_1+B_2, $B_1+(-B_2)$ and RB_1 is congruent (mod. $\mathfrak{A}$) to one of the set considered. Hence the set is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$.

If, conversely, a set Γ of elements

[1] The term 'quotient ring' and the symbol $\frac{\mathfrak{R}}{\mathfrak{M}}$ are adopted, as in the case of groups, from the view that both come from the same idea congruence. If $\mathfrak{M}$ is the ideal consisting of the element 0 alone, the quotient $\frac{\mathfrak{R}}{\mathfrak{M}}$ coincides with the original $\mathfrak{R}$.

$$B,\ B',\ B'',\ \ldots\ldots \pmod{\mathfrak{A}}$$

of $\frac{\mathfrak{R}}{\mathfrak{A}}$ be an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$, the elements of the co-sets

$$\mathfrak{A}+B,\ \mathfrak{A}+B',\ \mathfrak{A}+B'',\ \ldots\ldots$$

of $\mathfrak{R}$ as regards $\mathfrak{A}$, taken together, constitute an ideal of $\mathfrak{R}$. For, take any two elements, say $A+B$ and $A'+B'$, from the co-sets and an element R from $\mathfrak{R}$, and, since Γ is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$, we have

$$\begin{aligned}
&(A+B)+(A'+B') &&\equiv B+B' &&\equiv B_1 \pmod{\mathfrak{A}},\\
&(A+B)+\{-(A'+B')\} &&\equiv B+(-B') &&\equiv B_2 \pmod{\mathfrak{A}},\\
\text{and}\quad &R(A+B) &&\equiv RB &&\equiv B_3 \pmod{\mathfrak{A}},
\end{aligned}$$

where B_1, B_2, B_3 are certain elements of Γ. Therefore the elements $(A+B)+(A'+B')$, $(A+B)+\{-(A'+B')\}$ and $R(A+B)$ are contained in the co-sets $\mathfrak{A}+B_1$, $\mathfrak{A}+B_2$ and $\mathfrak{A}+B_3$ respectively, all of which belong to the said set of co-sets. Therefore the elements of the said co-sets, taken together, form an ideal of $\mathfrak{R}$, whose quotient as regards $\mathfrak{A}$ is Γ. Hence:

If $\mathfrak{B}$ is an ideal of $\mathfrak{R}$ which contains another $\mathfrak{A}$, $\frac{\mathfrak{B}}{\mathfrak{A}}$ is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$; and conversely if Γ is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$, there exists an ideal of $\mathfrak{R}$, whose quotient as regards $\mathfrak{A}$ is Γ.

From this follows:

Let $\mathfrak{A}$ be an ideal of a ring $\mathfrak{R}$. Then $\mathfrak{R}$ does or does not possess an ideal which contains $\mathfrak{A}$ and is distinct from both $\mathfrak{A}$ and $\mathfrak{R}$, according as the quotient ring $\frac{\mathfrak{R}}{\mathfrak{A}}$ is composite or simple.

Definition. If $\mathfrak{A}$, an ideal of a ring $\mathfrak{R}$, is such that there is no ideal of $\mathfrak{R}$, distinct from $\mathfrak{R}$, which contains $\mathfrak{A}$ and is distinct from $\mathfrak{A}$, then $\mathfrak{A}$ is called a *maximal* ideal of $\mathfrak{R}$.

If the norm of an ideal $\mathfrak{A}$ under $\mathfrak{R}$ is finite, there exists at least one maximal ideal of $\mathfrak{R}$.

§ 10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals of a ring $\mathfrak{R}$. The set of all the elements which are expressed in the form $A+B$, where A and B mean elements of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, is easily shown to be an ideal of $\mathfrak{R}$; this ideal derived from $\mathfrak{A}$ and $\mathfrak{B}$ is expressed by the symbol $(\mathfrak{A}, \mathfrak{B})$, as used commonly. The ideal $(\mathfrak{A}, \mathfrak{B})$ contains $\mathfrak{A}$ and $\mathfrak{B}$, and is contained in every ideal which contains both $\mathfrak{A}$ and $\mathfrak{B}$.

Next, if $\mathfrak{D}$ be the set of the elements common to $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{D}$

is also an ideal of $\mathfrak{R}$, which is called the *cross-cut* of $\mathfrak{A}$ and $\mathfrak{B}$. Indeed, if D be an element of $\mathfrak{D}$, the elements $-D$ and RD, where R is an element of R, belong to $\mathfrak{D}$, since D belongs to both $\mathfrak{A}$ and $\mathfrak{B}$.

Now let A and A_1 be two elements of the ideal $\mathfrak{A}$, and B and B_1 two of the ideal $\mathfrak{B}$. If

$$A+B \equiv A_1+B_1 \pmod{\mathfrak{D}},$$

then

$$A+(-A_1) \equiv B_1+(-B) \pmod{\mathfrak{D}}.$$

Hence $A+(-A_1)$ is an element of $\mathfrak{B}$, while it belongs to $\mathfrak{A}$; and consequently we have

$$A+(-A_1) \equiv 0 \pmod{\mathfrak{D}},$$

or

$$A \equiv A_1 \pmod{\mathfrak{D}},$$

and similarly

$$B \equiv B_1 \pmod{\mathfrak{D}}.$$

Thus from

$$A+B \equiv A_1+B_1 \pmod{\mathfrak{D}}$$

it follows that

$$A \equiv A_1 \pmod{\mathfrak{D}},$$

and

$$B \equiv B_1 \pmod{\mathfrak{D}};$$

and of this the converse is evidently true. Therefore *the elements of the form* $A+B$, *where* A *and* B *run over complete sets of incongruent* (*mod.* $\mathfrak{D}$) *elements of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *respectively, form a complete set of incongruent* (*mod.* $\mathfrak{D}$) *elements of* $(\mathfrak{A}, \mathfrak{B})$, *which is the quotient* $\frac{(\mathfrak{A}, \mathfrak{B})}{\mathfrak{D}}$.

Composition-Series of a Ring.

§ 11. Definition. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings. If a correspondence can be established between the elements of $\mathfrak{R}$ and those of $\mathfrak{R}'$, so that, if A and B be two elements of $\mathfrak{R}$ and A' and B' the two corresponding elements of $\mathfrak{R}'$, then (i) A' and B' are equal or unequal to each other, according as A and B are equal or unequal, (ii) to the sum $A+B$ there corresponds the sum $A'+B'$ and (iii) to the product AB there corresponds the product $A'B'$, then the two rings $\mathfrak{R}$ and $\mathfrak{R}'$ are said to be *isomorphic*, or of the same *type*.

THEOREM: *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are two ideals of a ring and if* $\mathfrak{D}$ *is the cross-cut of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *the quotient rings* $(\mathfrak{A}, \mathfrak{B})/\mathfrak{B}$ *and* $\mathfrak{A}/\mathfrak{D}$ *are of the same type, as also are the quotient rings* $(\mathfrak{A}, \mathfrak{B})/\mathfrak{A}$ *and* $\mathfrak{B}/\mathfrak{D}$.

When we take the elements of $(\mathfrak{A}, \mathfrak{B})$ with respect to the modulus $\mathfrak{B}$, the elements which become $\equiv 0 \pmod{\mathfrak{B}}$ are the elements of $\mathfrak{B}$, and those of $\mathfrak{A}$ which belong to $\mathfrak{D}$. If two elements A_1 and A_2 of $\mathfrak{A}$ are incongruent (mod. $\mathfrak{D}$), then these are also incongruent (mod. $\mathfrak{B}$), when A_1 and A_2 are considered for elements of $(\mathfrak{A}, \mathfrak{B})$. For, if it were true that $A_1 \equiv A_2 \pmod{\mathfrak{B}}$, then would

$$A_1+(-A_2) \equiv 0 \pmod{\mathfrak{B}},$$

i.e. $A_1+(-A_2)$ would be an element of $\mathfrak{B}$, while belonging to $\mathfrak{A}$. Hence $A_1+(-A_2)$ would belong to $\mathfrak{D}$, i.e. $A_1 \equiv A_2 \pmod{\mathfrak{D}}$; but this contradicts the assumption that A_1 and A_2 are incongruent (mod. $\mathfrak{D}$). Therefore, now, let

$$A_1,\ A_2,\ A_3,\ \ldots\ldots \pmod{\mathfrak{D}}$$

be a complete set of incongruent (mod. $\mathfrak{D}$) elements of $\mathfrak{A}$, which gives the ring $\mathfrak{A}/\mathfrak{D}$, then the elements

$$A_1,\ A_2,\ A_3,\ \ldots\ldots \pmod{\mathfrak{B}},$$

being considered for elements of $(\mathfrak{A}, \mathfrak{B})$ and being taken with respect to the modulus $\mathfrak{B}$, give a complete set of incongruent (mod. $\mathfrak{B}$) elements of $(\mathfrak{A}, \mathfrak{B})$, which is the ring $(\mathfrak{A}, \mathfrak{B})/\mathfrak{B}$. And moreover if

$$A_1+A_2 \equiv A_3 \pmod{\mathfrak{B}},$$

or

$$A_1A_2 \equiv A_3 \pmod{\mathfrak{B}},$$

then, as shown already, we have

$$A_1+A_2 \equiv A_3 \pmod{\mathfrak{D}},$$

or

$$A_1A_2 \equiv A_3 \pmod{\mathfrak{D}}$$

respectively. Therefore, by establishing a correspondence between the elements of $(\mathfrak{A}, \mathfrak{B})/\mathfrak{B}$ and those of $\mathfrak{A}/\mathfrak{D}$, so that to $A_1, A_2, A_3, \ldots\ldots \pmod{\mathfrak{B}}$ of the former there correspond respectively $A_1, A_2, A_3, \ldots\ldots \pmod{\mathfrak{D}}$ of the latter, we can see that these two rings are of the the same type. Similarly the rings $(\mathfrak{A}, \mathfrak{B})/\mathfrak{A}$ and $\mathfrak{B}/\mathfrak{D}$ can be treated.

Cor. I. If $\mathfrak{A}$ and $\mathfrak{B}$ are two distinct maximal ideals of a ring $\mathfrak{R}$, and if $\mathfrak{D}$ is the cross-cut of $\mathfrak{A}$ and $\mathfrak{B}$, then the quotient rings $\mathfrak{R}/\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{D}$ are of the same type, as also are the quotient rings $\mathfrak{R}/\mathfrak{A}$ and $\mathfrak{B}/\mathfrak{D}$; and consequently $\mathfrak{D}$ is a maximal ideal of both $\mathfrak{A}$ and $\mathfrak{B}$. (cf. §9).

Cor. 2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two distinct ideals of a ring $\mathfrak{R}$, and $\mathfrak{D}$ their cross-cut. If there is no ideal of $\mathfrak{R}$ which is contained in $\mathfrak{A}$ and contains $\mathfrak{D}$, there is no ideal of $\mathfrak{R}$ which is contained in $(\mathfrak{A}, \mathfrak{B})$ and contains $\mathfrak{B}$.

For, if there were an ideal $\mathfrak{C}$ of $\mathfrak{R}$, which was contained in $(\mathfrak{A}, \mathfrak{B})$ and contained $\mathfrak{B}$, $\mathfrak{C}$ would contain elements of $\mathfrak{A}$ which do not belong to $\mathfrak{B}$; but not all of the elements of $\mathfrak{A}$ would be contained in $\mathfrak{C}$. Hence the cross-cut of two ideals $\mathfrak{A}$ and $\mathfrak{C}$ of $\mathfrak{R}$ would be larger than $\mathfrak{D}$ and distinct from $\mathfrak{A}$. This contradicts the assumption that $\mathfrak{A}$ contains no ideal of $\mathfrak{R}$ which contains $\mathfrak{D}$.

§ 12. Let $\mathfrak{A}_1$ be a maximal ideal of a given ring $\mathfrak{R}$, $\mathfrak{A}_2$ a maximal ideal of $\mathfrak{A}_1$, and so on.

Definitions. When the series of rings

$$(A) \qquad \mathfrak{R}, \mathfrak{A}_1, \mathfrak{A}_2, \ldots\ldots$$

can be obtained in the manner just described, the series (A) is called a *composition-series*[1] of $\mathfrak{R}$.

The set of rings

$$\frac{\mathfrak{R}}{\mathfrak{A}_1}, \frac{\mathfrak{A}_1}{\mathfrak{A}_2}, \ldots\ldots$$

is called a set of quotient rings of $\mathfrak{R}$ derived from the composition-series (A).

Take the first n terms from the series (A). And the series

$$\mathfrak{R}, \mathfrak{A}_1, \mathfrak{A}_2, \ldots\ldots, \mathfrak{A}_{n-1}$$

thus obtained is called a composition-series of $\mathfrak{R}$ which has $\mathfrak{A}_{n-1}$ as the last term.

A ring under which the norm of each ideal is finite, such as a ring generated by an algebraic integer, has always a composition-series; and in this case each of the quotient rings is finite in number of elements.

It may happen that a ring has more than one distinct composition-series; but we can prove the

THEOREM[2]: *Any two composition-series of a ring, whose last terms are the same, consist of the same number of terms and lead to two sets*

---

1 The series is named a composition-series as in the case of groups.

2 This is analogous to the important theorem on composition-series of a group, due to Hölder, but somewhat generalized. Cf. Burnside, Theory of Groups of Finite Order, 2. de., pp. 64-69.

of quotient rings, which are identical with each other except as regards the sequence in which they occur.

Let us suppose that, if the number of terms of one of two composition-series of a given ring, whose last terms are the same, is not greater than a given number n, it has already been shown that these two series contain the same number of terms and that the quotient rings derived from them are identical except as regards their sequence. If a ring $\mathfrak{R}$ has two composition-series having the same last term, one of which consists of $n+1$ terms, let them be

(1) $$\mathfrak{R},\ \mathfrak{A}_1,\ \mathfrak{A}_2,\ \ldots\ldots,\ \mathfrak{A}_n,$$

and

(2) $$\mathfrak{R},\ \mathfrak{B}_1,\ \mathfrak{B}_2,\ \ldots\ldots,\ \mathfrak{B}_m,$$

where $\mathfrak{B}_m$ is identical with $\mathfrak{A}_n$, and $m \gtreqless n$. If $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are identical, it evidently follows, from the supposition, that $m=n$ and that the two sets of quotient rings derived from them are identical except as regards their sequence. If on the contrary $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are distinct, let $\mathfrak{D}$ be the cross-cut of $\mathfrak{A}_1$ and $\mathfrak{B}_1$. Then $\mathfrak{D}$ is a maximal ideal of both $\mathfrak{A}_1$ and $\mathfrak{B}_1$ (§ 11, cor. 1). And $\mathfrak{D}$ contains $\mathfrak{A}_n$; because $\mathfrak{A}_1$ and $\mathfrak{B}_1$ contain $\mathfrak{A}_n$ and $\mathfrak{B}_m$ respectively, while $\mathfrak{A}_n$ and $\mathfrak{B}_m$ are identical. If $\mathfrak{D}$ coincides with $\mathfrak{A}_2$, $\mathfrak{D}$ has a composition-series consisting of $n-1$ terms, having the last term $\mathfrak{A}_n$. If $\mathfrak{D}$ does not coincide with $\mathfrak{A}_2$, let $\mathfrak{D}_1$ be the cross-cut of $\mathfrak{D}$ and $\mathfrak{A}_2$. Then $\mathfrak{D}_1$ is a maximal ideal of both $\mathfrak{D}$ and $\mathfrak{A}_2$, and contains $\mathfrak{A}_n$. If $\mathfrak{D}_1$ coincides with $\mathfrak{A}_3$, $\mathfrak{D}$ has a composition-series containing $n-1$ terms, with the last term $\mathfrak{A}_n$. If not, take $\mathfrak{D}_2$ as the cross-cut of $\mathfrak{D}_1$ and $\mathfrak{A}_3$ Then $\mathfrak{D}_2$ is a maximal ideal of both $\mathfrak{D}_1$ and $\mathfrak{A}_3$, and contains $\mathfrak{A}_n$. If $\mathfrak{D}_2$ does not yet coincide with $\mathfrak{A}_4$, continue this process. Since the number of terms of the series (1) is finite, it can be shown that, after a finite number of repetitions, we reach a composition-series

$$\mathfrak{D},\ \mathfrak{D}_1,\ \mathfrak{D}_2,\ \ldots\ldots,\ \mathfrak{D}_{n-2}\ (\mathfrak{D}_{n-2}=\mathfrak{A}_n)$$

of $\mathfrak{D}$ which has $\mathfrak{A}_n$ as the last term; and that the series

(3) $$\mathfrak{R},\ \mathfrak{A}_1,\ \mathfrak{D},\ \mathfrak{D}_1,\ \ldots\ldots,\ \mathfrak{D}_{n-2}\ (\mathfrak{D}_{n-2}=\mathfrak{A}_n),$$

and

(4) $$\mathfrak{R},\ \mathfrak{B}_1,\ \mathfrak{D},\ \mathfrak{D}_1,\ \ldots\ldots,\ \mathfrak{D}_{n-2}\ (\mathfrak{D}_{n-2}=\mathfrak{A}_n)$$

are two composition-series of $\mathfrak{R}$, which have the same last term, con-

tain the same number of terms and give the same quotient rings. For it has been shown in § 11, cor. 1 that $\mathfrak{R}/\mathfrak{A}_1$ and $\mathfrak{B}_1/\mathfrak{D}$ are of the same type, as also are $\mathfrak{R}/\mathfrak{B}_1$ and $\mathfrak{A}_1/\mathfrak{D}$. But the two composition-series

$$\mathfrak{A}_1,\ \mathfrak{A}_2,\ \mathfrak{A}_3,\ \ldots\ldots,\ \mathfrak{A}_n,$$

and

$$\mathfrak{A}_1,\ \mathfrak{D},\ \mathfrak{D}_1,\ \ldots\ldots,\ \mathfrak{D}_{n-2}\ (\mathfrak{D}_{n-2} = \mathfrak{A}_n)$$

of $\mathfrak{A}_1$ have the same last term $\mathfrak{A}_n$, and the former contains just n terms. Hence they give the same quotient rings by supposition. Since the composition-series

$$\mathfrak{B}_1,\ \mathfrak{B}_2,\ \ldots\ldots,\ \mathfrak{B}_m \quad (\mathfrak{B}_m = \mathfrak{A}_n),$$

and

$$\mathfrak{B}_1,\ \mathfrak{D},\ \mathfrak{D}_1,\ \ldots\ldots\ \mathfrak{D}_{n-2} \quad (\mathfrak{D}_{n-2} = \mathfrak{A}_n)$$

of $\mathfrak{B}_1$ also have the same last term, and the latter contains n terms, they by supposition contain the same number of terms and give the same quotient rings. Hence the series (1) and (3) must contain the same number of terms and give the same quotient rings, as also do the series (2) and (4), while the same holds for the series (3) and (4). The theorem, therefore, is also true when the number of terms of one of two composition-series of a ring is $n+1$. The theorem is true when $n=3$. For let

$$\mathfrak{R},\ \mathfrak{A}_1,\ \mathfrak{A}_2,$$

and

$$\mathfrak{R},\ \mathfrak{B}_1,\ \mathfrak{B}_2,\ \ldots\ldots$$

be two composition-series of $\mathfrak{R}$ which have $\mathfrak{A}_2$ as the last terms. If $\mathfrak{B}_1$ coincides with $\mathfrak{A}_1$, $\mathfrak{B}_2$ must be identical with $\mathfrak{A}_2$; because $\mathfrak{B}_2$ contains the maximal ideal $\mathfrak{A}_2$ of $\mathfrak{A}_1$. Hence we need only consider the case where $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are distinct. The cross-cut of $\mathfrak{A}_1$ and $\mathfrak{B}_1$ is a maximal ideal of both $\mathfrak{A}_1$ and $\mathfrak{B}_1$, and moreover contains $\mathfrak{A}_2$; hence the cross-cut must coincide with $\mathfrak{A}_2$ and also with $\mathfrak{B}_2$, since $\mathfrak{B}_2$ is a maximal ideal of $\mathfrak{B}_1$, which contains $\mathfrak{A}_2$. Therefore both the series contain three terms, and have $\mathfrak{A}_2$, which is the cross-cut of $\mathfrak{A}_1$ and $\mathfrak{B}_1$, as their last terms; and consequently give the same quotient rings (by § 11, cor. 1). The theorem is therefore universally true.

§ 13. Definition. Let

$$\mathfrak{R},\ \mathfrak{A}_1,\ \mathfrak{A}_2,\ \ldots\ldots$$

be a series of ideals of a ring $\mathfrak{R}$, in which each ideal is contained in the preceding one, while there is no ideal of $\mathfrak{R}$ contained in any single

ideal of the series also containing the next ideal. The series, if it exists, shall be called a *chief-composition-series* of $\mathfrak{R}$.

The series

$$\mathfrak{R},\ \mathfrak{A}_1,\ \mathfrak{A}_2,\ \ldots\ldots,\ \mathfrak{A}_{n-1},$$

which consists of the first n terms of the said chief-composition-series of $\mathfrak{R}$, is called a chief-composition-series of $\mathfrak{R}$ which has $\mathfrak{A}_{n-1}$ as the last term.

THEOREM: *Any two chief-composition-series of a ring, whose last terms are the same, consist of the same number of terms and lead to two sets of quotient rings, which are identical with each other except as regards the sequence in which they occur.*

Let us suppose that, when the number of terms of one of two chief-composition-series of a given ring is not greater than a given number n, the theorem holds true. If a ring $\mathfrak{R}$ has two chief-composition-series which have the same last terms and one of which is of $(n+1)$ terms, let them be

$$\mathfrak{R},\ \mathfrak{A}_1,\ \ldots\ldots,\ \mathfrak{A}_{n-1},\ \mathfrak{A}_n,$$

and

$$\mathfrak{R},\ \mathfrak{B}_1,\ \ldots\ldots,\ \mathfrak{B}_{m-1},\ \mathfrak{B}_m,$$

where $\mathfrak{B}_m$ is identical with $\mathfrak{A}_n$, and $m \geqq n$. If $\mathfrak{A}_{n-1}$ and $\mathfrak{B}_{m-1}$ are identical, the theorem holds true for these series by our supposition. If not, take the ideal $(\mathfrak{A}_{n-1}, \mathfrak{B}_{m-1})$ of $\mathfrak{R}$. Then we see by § 11, cor. 2 that there is no ideal of $\mathfrak{R}$ which is contained in $(\mathfrak{A}_{n-1}, \mathfrak{B}_{m-1})$ and contains $\mathfrak{A}_{n-1}$ or $\mathfrak{B}_{m-1}$. If $(\mathfrak{A}_{n-1}, \mathfrak{B}_{m-1})$ is distinct from $\mathfrak{A}_{n-2}$, this process is repeated. By proceeding thus in reverse order, as in the proof of the last theorem, we have a chief-composition-series of $\mathfrak{R}$ which has $(\mathfrak{A}_{n-1}, \mathfrak{B}_{m-1})$ as its last term. And it is shown similarly that the theorem is also true when the number of terms of one of the series is $n+1$, and moreover that it is true for $n=3$. Therefore it is universally true.

Simple Rings.

§ 14. Let $\mathfrak{R}$ be a ring and A a given element of $\mathfrak{R}$. *The set* $\mathfrak{M}$ *of all the elements* X *for which* $AX=0$ *is an ideal of* $\mathfrak{R}$. Indeed, if X_1 and X_2 are two elements of $\mathfrak{M}$ and if R is an element of $\mathfrak{R}$, we have

$$A(-X_1) = 0,$$

$$A(X_1+X_2) = AX_1+AX_2 = 0,$$

and $$A \cdot X_1 R = 0,$$

which show that $\mathfrak{M}$ is an ideal of $\mathfrak{R}$.

In case $\mathfrak{R}$ is simple, $\mathfrak{M}$ must be either $\mathfrak{R}$ itself or the ideal consisting of the element 0 alone. Therefore if $A=0$, $\mathfrak{M}$ evidently coincides with $\mathfrak{R}$; but if $A \neq 0$, the equation $AX=0$ as regards X is satisfied either by the element 0 only or by each element of $\mathfrak{R}$. If for each element A, except 0, the equation $AX=0$ as regards X has only one root $X=0$, then the product BC of two elements B and C of $\mathfrak{R}$ is equal to 0 when, and only when, at least one of B and C is equal to 0. Suppose, on the contrary, that for a certain element A, not equal to 0, the equation $AX=0$ is satisfied by each element of $\mathfrak{R}$. Take any two elements B and C from $\mathfrak{R}$. If $BC \neq 0$, then the ideal of $\mathfrak{R}$, which consists of all the roots of the equation $BX=0$, contains 0 and A, but not C; consequently $\mathfrak{R}$ can not be simple. Therefore if $\mathfrak{R}$ is simple, the product of any two elements of $\mathfrak{R}$ must be equal to 0, under the above assumption. Thus if the ring $\mathfrak{R}$ is simple, there occur the two cases:

(i) The product of two elements of $\mathfrak{R}$ is equal to 0 when, and only when, at least one of the factors is equal to 0;

(ii) The product of any two elements is equal to 0.

Considering the first case, let A and B be two elements of $\mathfrak{R}$, and $A \neq 0$. If there is in $\mathfrak{R}$ no element X such that $AX=B$, the set of the elements C, for which the equation $AX=C$ has at least one root, must be an ideal of $\mathfrak{R}$, which is distinct from both $\mathfrak{R}$ itself and the ideal consisting of the element 0 alone. But, since $\mathfrak{R}$ is simple, the equation $AX=B$ as regards X must have a root. Therefore the postulates (i) and (ii) of § 1 hold, and $\mathfrak{R}$ must be a field. It has been shown already in § 8 that a field is a simple ring.

Considering the second case, take an element V not equal to 0 from $\mathfrak{R}$. If the sum of a finite number of V is never equal to 0, the set

$$0, \pm V, \pm 2V, \ldots\ldots,$$

where nV means V taken n times, must be an ideal of $\mathfrak{R}$ and must coincide with $\mathfrak{R}$, because of the simpleness of $\mathfrak{R}$. But in this case $\mathfrak{R}$ can not be simple, since it contains the ideal consisting of the elements

$$0, \pm 2V, \pm 4V, \ldots\ldots.$$

Therefore the sum of a certain number n of V must be equal to 0, i.e. $nV=0$, and $\mathfrak{R}$ consists of

$$0,\ V,\ 2V,\ \ldots\ldots,\ (n-1)\,V.$$

If n is composite and p a factor of n, $\mathfrak{R}$ contains the ideal consisting of the elements

$$0,\ pV,\ 2pV,\ \ldots\ldots,\ \left(\frac{n}{p}-1\right)pV,$$

and can not be simple. But, since $\mathfrak{R}$ is supposed to be simple, n must be a prime. If, conversely, n is a prime, the ring consisting of

$$0,\ V,\ 2V,\ \ldots\ldots,\ (n-1)\,V$$

is simple; because an ideal of a ring is a subring of the ring and the number of the elements of a finite ring is divisible by that of its subring (§ 6). Hence we have the

THEOREM: *A simple ring is either a field or one which consists of p elements*

$$0,\ V,\ 2V,\ \ldots\ldots,\ (p-1)\,V,$$

where p is a prime, $pV=0$ but $iV \neq 0$ when $0<i<p$, and $V^2=0$; and conversely.

Let us, further, consider an element V for which $V^2=0$[1]. Then the elements

$$0,\ \pm V,\ \pm 2V,\ \ldots\ldots,$$

where nV means V taken n times, constitute a ring, which we denote by $[V]$. And the elements

$$0,\ \pm nV,\ \pm 2nV,\ \ldots\ldots$$

constitute an ideal of $[V]$, which we denote by $[nV]$. The quotient ring $[V]/[nV]$ consists of

$$0,\ V,\ 2V,\ \ldots\ldots,\ (n-1)\,V \ (\text{mod. } [nV]);$$

and, if n is a prime, it is simple, as stated before.

§ 15. If $\mathfrak{A}$ is a maximal ideal of a ring $\mathfrak{R}$, the quotient ring $\frac{\mathfrak{R}}{\mathfrak{A}}$ is simple, as shown before; and consequently $\frac{\mathfrak{R}}{\mathfrak{A}}$ is either a field or a simple ring of the type $\frac{[V]}{[pV]}$. When $\mathfrak{R}$ contains especially an element U by which every element of $\mathfrak{R}$ multiplied is equal to itself, the quotient ring $\frac{\mathfrak{R}}{\mathfrak{A}}$ as regards the maximal ideal $\mathfrak{A}$ must be a field, because $U^2=U$, and U can not belong to $\mathfrak{A}$.

[1] Such as let V be a vector and let V^2 denote the vector-square.

Minimal Ideals. Types of the Quotient Rings of a Chief-Composition-Series.

§ 16. An ideal $\mathfrak{A}$ of a ring $\mathfrak{R}$ is said to be *minimal* when there is no ideal of $\mathfrak{R}$ which is contained in $\mathfrak{A}$ and is distinct from both $\mathfrak{A}$ and the ideal[1] consisting of the element 0 alone.

Let $\mathfrak{A}$ be a subring of a ring $\mathfrak{R}$ and R an element of $\mathfrak{R}$. We denote by $R\mathfrak{A}$ the set of all the products formed by multiplying R into each element of $\mathfrak{A}$. Then, if $\mathfrak{A}$ is an ideal of $\mathfrak{R}$, $R\mathfrak{A}$ must be also an ideal of $\mathfrak{R}$, which is contained in $\mathfrak{A}$; and, if the ideal $\mathfrak{A}$ is minimal, $R\mathfrak{A}$ is either coincident with $\mathfrak{A}$ or the 0-ideal.

When an element R of $\mathfrak{R}$ is given, the elements X of the ideal $\mathfrak{A}$ of $\mathfrak{R}$, for which $RX=0$, evidently constitute an ideal of $\mathfrak{R}$, which is contained in $\mathfrak{A}$. Therefore, when $\mathfrak{A}$ is minimal, if an element R of $\mathfrak{R}$ multiplied by a certain one, not equal to the element 0, of $\mathfrak{A}$ gives 0, the element R multiplied by one of $\mathfrak{A}$ must give 0. In other words, the elements S of $\mathfrak{R}$, for which $S\mathfrak{A}=0$, form an ideal $\mathfrak{S}$ of $\mathfrak{R}$; and, if $\mathfrak{A}$ is minimal, the product SA, A being an element contained in $\mathfrak{A}$ and not equal to 0, is equal to 0 when, and only when, $S \equiv 0 \pmod{\mathfrak{S}}$.

§ 17. Proceeding to investigate the type of a minimal ideal of a ring, let $\mathfrak{A}$ be a minimal ideal of a ring $\mathfrak{R}$. Then there occur the following three cases to be considered: (i) there exists in $\mathfrak{A}$ at least one element A such that $A\mathfrak{A}$ coincides with $\mathfrak{A}$; (ii) for every element R of $\mathfrak{R}$, $R\mathfrak{A}$ is the 0-ideal; (iii) there exists in $\mathfrak{R}$ at least one element R for which $R\mathfrak{A}=\mathfrak{A}$, but $\mathfrak{A}$ multiplied by every element of it gives the 0-ideal.

Considering the first case, if A_1 and A_2 are two elements of $\mathfrak{A}$ such that $A_1\mathfrak{A}=0$ and $A_2\mathfrak{A}=0$, then $\{A_1+(\pm A_2)\}\mathfrak{A}=0$, and $RA_1\mathfrak{A}=0$, where R is an element of $\mathfrak{R}$; and moreover $A_1+(\pm A_2)$ and RA_1 both belong to $\mathfrak{A}$. Therefore the elements X of $\mathfrak{A}$ for which $X\mathfrak{A}=0$ form an ideal of $\mathfrak{R}$ which is contained in $\mathfrak{A}$; and must be either the 0-ideal or $\mathfrak{A}$ itself, because $\mathfrak{A}$ is minimal. But, since there is assumed the existence of an element of $\mathfrak{A}$ by which $\mathfrak{A}$ multiplied coincides with itself, the said ideal must be the 0-ideal. Hence for every element A, except 0, of $\mathfrak{A}$ the ideal $A\mathfrak{A}$ coincides with $\mathfrak{A}$. Consequently for any two elements A and B of $\mathfrak{A}$, provided that A is not equal to 0, there exists in $\mathfrak{A}$ at least one element X such that $AX=B$; showing that $\mathfrak{A}$ is a field.

[1] Hereafter the ideal consisting of the element 0 alone will be called the 0-ideal.

In the second case the products of every two elements of $\mathfrak{A}$ are all equal to 0, and every ideal of $\mathfrak{A}$ is also an ideal of $\mathfrak{R}$. Hence $\mathfrak{A}$ must be a simple ring in which every product of its elements is equal to 0; consequently $\mathfrak{A}$ is a finite simple ring of the type

$$0,\ V,\ 2V,\ \ldots\ldots,\ (p-1)\,V,$$

where $V^2 = 0$, $pV = 0$ but $iV \neq 0\ (0 < i < p)$, and p is a prime (§ 14).

Lastly, in the third case every product of elements of $\mathfrak{A}$ is equal to 0; and there are again to be considered two cases: there exists or there does not exist in $\mathfrak{A}$ at least one element, not equal to 0, such that it, taken a finite number of times, is equal to 0. If such elements exist, there must evidently exist at least one element which, taken a prime number of times, makes 0. Let it be V, i.e. $pV=0$, but $iV \neq 0\ (0 < i < p)$, where p is a prime and pV means V taken p times. Then the p elements

$$0,\ V,\ 2V,\ \ldots,\ (p-1)\,V$$

form an ideal $\mathfrak{B}$ of $\mathfrak{A}$. If an element R of $\mathfrak{R}$, for which $RV \neq 0$, is taken, the product $R(iV)$, $(0 < i < p)$, is not equal to 0 (§ 16), and the set $R\mathfrak{B}$ is also an ideal of $\mathfrak{A}$, which either coincides with $\mathfrak{B}$ or has no element, except 0, common to $\mathfrak{B}$. Hence by multiplying $\mathfrak{B}$ by each element of $\mathfrak{R}$ we obtain a finite or infinite number of distinct ideals of $\mathfrak{A}$, all of which are of the same type as $\mathfrak{B}$, and every two of which have no common element other than 0. They, taken together, produce an ideal of $\mathfrak{R}$ contained in $\mathfrak{A}$; consequently it must coincides with $\mathfrak{A}$. Therefore, if there exists in $\mathfrak{A}$ at least one element, unequal to 0, such that it, taken a finite number of times, makes 0, the minimal ideal $\mathfrak{A}$ is what is derived from a finite or infinite number of rings of the same type as the said ring $\mathfrak{B}$. Hence we have the

THEOREM: *A minimal ideal of a ring is one of the following four: (i) a field; (ii) a finite simple ring of the type*

$$0,\ V,\ 2V,\ \ldots,\ (p-1)\,V,$$

where $V^2=0$, $pV=0$, and p is prime; (iii) a ring such that every element of it multiplied by any one of it gives 0, and it is derived from a finite or infinite number of rings of the type (ii); (iv) a ring such that every element of it multiplied by any one of it gives 0, but no element taken a finite number of times is equal to 0.

§ 18. Now every field of finite order, according to Prof. Dickson's study[1], is of order p^n, p being a prime, and may be represented as a Galois field of order p^n; and a Galois field is defined uniquely by its order. If, in the case of the type (iii) in the last theorem, the minimal ideal is derived from n rings, it contains p^n unequal elements, and is an Abelian group of order p^n and type (1, 1, 1, ..., with n units) with respect to addition[2]; and consequently it is also uniquely determined by the number of elements. Thus *the type of a minimal ideal of a ring consisting of a finite number of elements is determined uniquely by the number of elements in each case, a field or not.*

§ 19. An example for case (iv). Take the set of pairs (a, b) of real numbers; define addition and multiplication among them as

$$(a, b)+(a', b') = (a+a', b+b')$$

and

$$(a, b)(a', b') = (aa', ab' + a'b),$$

and we have a ring of elements (a, b). The elements $(0, b)$ form a minimal ideal of this ring, which is of type (iv).

§ 20. Let $\mathfrak{A}$, $\mathfrak{B}$ be two distinct ideals of a ring $\mathfrak{R}$; assume that $\mathfrak{B}$ contains $\mathfrak{A}$. We have shown in § 9 that the quotient ring $\frac{\mathfrak{B}}{\mathfrak{A}}$ is an ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$ and that to an ideal Γ of $\frac{\mathfrak{R}}{\mathfrak{A}}$, there corresponds the ideal of $\mathfrak{R}$ whose quotient as regards $\mathfrak{A}$ is Γ. If Γ is contained in $\frac{\mathfrak{B}}{\mathfrak{A}}$, the corresponding ideal of $\mathfrak{R}$ is evidently contained in $\mathfrak{B}$. Therefore, if there is no ideal of $\mathfrak{R}$ contained in $\mathfrak{B}$ and containing $\mathfrak{A}$, $\frac{\mathfrak{B}}{\mathfrak{A}}$ must be a minimal ideal of $\frac{\mathfrak{R}}{\mathfrak{A}}$. Hence, *if $\mathfrak{A}_r$ and $\mathfrak{A}_{r+1}$ are two consecutive terms of a chief-composition-series of a ring, there are applicable the facts stated in §§ 17, 18 to the investigation of the type of the quotient ring* $\frac{\mathfrak{A}_r}{\mathfrak{A}_{r+1}}$.

For example, let $\mathfrak{r}$ be the set of the algebraic integers of an algebraic field, and $\mathfrak{p}$, $\mathfrak{q}$ two distinct prime ideals of the field. Then $\mathfrak{p}$ and $\mathfrak{q}$ both are maximal ideals of $\mathfrak{r}$; and there is no ideal of $\mathfrak{r}$ contained in $\mathfrak{p}$ and containing $\mathfrak{pq}$ or $\mathfrak{p}^2$, so that the quotient rings

---

1 L. E. Dickson, Linear Groups with an exposition of the Galois Field Theory, p. 14.

2 An Abelian group with respect to addition was called a *Modul* by Dedekind: Dirichlet-Dedekind, Vorlesungen über Zahlentheorie, 4 ed., § 168.

$\frac{\mathfrak{p}}{\mathfrak{pq}}$ and $\frac{\mathfrak{p}}{\mathfrak{p}^2}$ are minimal ideals of $\frac{\mathfrak{r}}{\mathfrak{pq}}$ and $\frac{\mathfrak{r}}{\mathfrak{p}^2}$ respectively. And $\frac{\mathfrak{p}}{\mathfrak{pq}}$ is a field, but $\frac{\mathfrak{p}}{\mathfrak{p}^2}$ not.

December, 1916.

Vol. II., No. 3, Published July 20th, 1917.

Price - - - Yen .80

These Memoirs are on sale at

MARUZEN COMPANY, LTD.

TOKYO, OSAKA, KYOTO, FUKUOKA & SENDAI.

大正六年七月十七日印刷

大正六年七月二十日發行

編纂兼發行者　京都帝國大學理科大學

印刷者　野村宗十郎

東京市京橋區築地三丁目十一番地

印刷所　株式會社　東京築地活版製造所

東京市京橋區築地二丁目十七番地

賣捌所　丸善株式會社

東京、大阪、京都、福岡、仙臺

CONTENTS.

PAGE

On the Solubility of Lead Acetate in Water.
By Yukichi Osaka and Reijiro Hara. - - - - - - - 147

Graphical Solution, etc.
By Takeo Wada. - - - - - - - - - - - - - 151

On the Absorption Spectrum of Oxyhœmoglobin in the Ultra-violet and Extreme Ultra-violet Region.
By Toshizo Mashimo. - - - - - - - - - - - 199

On Congruences.
By Masazo Sono. - - - - - - - - - - - - - 203